ELEMENTS

OF

GEOMETRY,

GEOMETRICAL ANALYSIS,

AND

PLANE TRIGONOMETRY.

WITH AN

APPENDIX,

AND COPIOUS NOTES AND ILLUSTRATIONS.

BY

JOHN LESLIE, F.R.S.E.
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY
OF EDINBURGH.

SECOND EDITION,

IMPROVED AND ENLARGED.

EDINBURGH:
PRINTED FOR JOHN BALLANTYNE AND COMPANY;
AND FOR
LONGMAN, HURST, REES, ORME AND BROWN,
LONDON.

1811.

June 12, 1914

[illegible]

PREFACE.

THE volume now laid before the public, is the first of a projected Course of Mathematical Science. Many compendiums or elementary treatises have appeared—at different times, and of various merit; but there seemed still wanting in our language, a work that should embrace the subject in its full extent,—that should unite theory with practice, and connect the ancient with the modern discoveries. The magnitude and difficulty of such a task might deter an individual from the attempt, if he were not deeply impressed with the importance of the undertaking, and felt his exertions to accomplish it animated by zeal, and supported by active perseverance.

The study of Mathematics holds forth two capital objects:—While it traces the beautiful relations of figure and quantity, it likewise accustoms the mind to the invaluable exercise of patient attention and accurate reasoning. Of these distinct objects, the last is perhaps the most important in a course of

liberal e. on. For this purpose, the geometry of the Greeks is the most powerfully recommended, as bearing the stamp of that acute people, and displaying the finest specimens of logical deduction. Some of the conclusions, indeed, might be reached by a sort of calculation; but such an artificial mode of procedure gives only an apparent facility, and leaves no clear or permanent impression on the mind.

We should form a wrong estimate, however, did we consider the Elements of Euclid, with all its merits, as a finished production. That admirable work was composed at the period when geometry was making its most rapid advances, and new prospects were opening on every side. No wonder that its structure should now appear loose and defective. In adapting it to the actual state of the science, I have therefore endeavoured carefully to retain the spirit of the original, but have sought to enlarge the basis, and to dispose the accumulated materials into a regular and more compact system. By simplifying the order of arrangement, I presume that I have materially abridged the labour of the student. The numerous additions which are incorporated in the text, so far from retarding, will rather facilitate his progress, by rendering more continous the chain of demonstration.

The view which I have given of the nature of Proportion, in the fifth Book, will, I hope, contribute to remove the chief difficulties attending that im-

portant subject. The sixth Book, which exhibits the application of the doctrine of ratios, contains a copious selection of propositions, not only beautiful in themselves, but which pave the way to the higher branches of Geometry, or lead immediately to valuable practical results. The Appendix, without claiming the same degree of utility, will not perhaps be deemed the least interesting portion of the volume, since the ingenious resources which it discloses for the construction of certain problems are calculated to afford a very pleasing and instructive exercise.

The part which has cost me the greatest pains, is that devoted to Geometrical Analysis. The first Book consists of a series of the choicest problems, rising above each other in gradual succession. The second and third Books are almost wholly occupied with the researches of the Ancient Analysis. In framing them, I have consulted a great diversity of authors, some of whom are of difficult access. The labour of condensing the scattered materials, will be duly estimated by those, who, taking delight in such fine speculations, are thus admitted at once to a rich and varied repast. The analytical investigations of the Greek geometers are indeed models of simplicity, clearness, and unrivalled elegance; and though miserably defaced by the riot of time and barbarism, they will yet be regarded by every person capable of appreciating their beauties, as among the noblest monuments of human genius. It is matter of deep regret, that Algebra, or the Modern Analysis, from

the mechanical facility of its operations, has contributed, especially on the Continent, to vitiate the taste and destroy the proper relish for the strictness and purity so conspicuous in the ancient method of demonstration. The study of geometrical analysis appears admirably fitted to improve the intellect, by training it to habits of precision, arrangement, and close application. If the taste thus acquired be not allowed to obtain undue ascendency, it may be transferred with eminent utility to Algebra, which, having shot up prematurely, wants reform in almost every department.

The Elements of Trigonometry are as ample as my plan would allow. I have explained fully the properties of the lines about the circle, and the calculation of the trigonometrical tables; nor have I omitted any proposition which has a distinct reference to practice. Some of the problems annexed are of essential consequence in marine surveying.

In the improvement of this edition, I have spared no trouble or expence. The whole has undergone a careful and minute revision; and by adopting a denser page, I have been able, without adding much to the size of the volume, yet greatly to augment the work. The text has been simplified and reduced to a shorter compass, by throwing such propositions as were less elementary to the Notes. Other Notes of a simpler kind, and marked by the reference in italics, are intended chiefly to engage the attention of the young student. In various parts, the demon-

strations are occasionally abridged. The books of Geometrical Analysis have been rendered more complete by numerous insertions, and particularly an abstract of all the different investigations left us by the ancients, relating to the trisection of an angle and the duplication of the cube. The Elements of Trigonometry are much expanded, and brought to include whatever appears most valuable in recent practice. But the greatest additions have been made in the Notes and Illustrations, which will be found, I should hope, to contain various and useful information. The more advanced student may peruse with advantage the historical and critical remarks; and some of the disquisitions, and the solutions of certain more difficult problems relative to trigonometry and geodesiacal operations, in which the modern analysis is sparingly introduced, are of a nature sufficiently interesting, perhaps, to claim the notice of proficients in science. I have simplified, and materially enlarged the formulæ connected with trigonometrical computation; explained the art of surveying, in its different branches; and given reduced plans, blended with the narrative of the great operations lately carried on both in England and France. I have likewise shown a very simple method for calculating heights from barometrical observations, accompanied by illustrative sections; and I have been thence led to state the law of climate, as it is modified by elevation. On this attractive subject, I should have dwelt with pleasure, had the limits of the volume permitted.

The plan now in contemplation might perhaps be comprised in five volumes. The next volume is intended to treat of the Geometry of curve lines, the intersections of planes, and the properties of solids, including the doctrine of the sphere and the calculation of spherical triangles, with the elements of perspective and projection. The third volume will be devoted to Algebra,—a wide and rank field, which still needs the most sedulous cultivation. As an introduction to that difficult task, I design to prepare, with all convenient speed, a short tract on the Principles of Arithmetic; a work much wanted in our ordinary course of education, and which, were it executed with taste, and in the spirit of philosophy, would be admirably fitted for opening the mind of the pupil. The fourth and fifth volumes will embrace the differential and integral calculus, with their principal applications. But to accomplish this vast undertaking would require years of health and inflexible resolution; and under all the discouragements which attend the publication of scientific works, I cannot look forward to its completion without feeling extreme solicitude.

It is the nature of mathematical science to advance in continual progression. Each step carries it to others still higher; new relations are descried; and the most distant objects seem gradually to approximate. But, while science thus enlarges its bounds, it likewise tends uniformly to simplicity and concentration. The discoveries of one age are, perhaps in the next, melted down into the mass of ele-

mentary truths. What are deemed at first merely objects of enlightened curiosity, become, in due time, subservient to the most important interests. Theory soon descends to guide and assist the operations of practice. To the geometrical speculations of the Greeks, we may distinctly trace whatever progress the moderns have been enabled to achieve in mechanics, navigation, and the various complicated arts of life. A refined analysis has unfolded the harmony of the celestial motions, and conducted the philosopher, through a maze of intricate phenomena, to the great laws appointed for the government of the Universe.

College of Edinburgh,
September 2. 1811.

ELEMENTS

OF

GEOMETRY.

GEOMETRY is that branch of natural science which treats of figured space.

Our knowledge concerning external objects is grounded entirely on the information received through the medium of the senses. The science of physics considers bodies as they actually exist, invested at once with all their various qualities, and endued with their peculiar affections. Its researches are hence directed by that refined species of observation which is termed Experiment. Geometry takes a more limited view, and, selecting only the generic property of *magnitude*, it can, from the extreme simplicity of its basis, safely pursue the most lengthened train of investigation, and arrive with perfect certainty at the remotest conclusion. It contemplates merely the forms which bodies present, and the spaces which they occupy. Geometry is thus likewise founded on external observation; but such observation is

so familiar and obvious, that the primary notions which it furnishes might seem intuitive, and have often been regarded as innate. This science is therefore super-eminently distinguished by the luminous evidence which constantly attends every step of its progress.

PRINCIPLES.

In contemplating an external object, we can, by successive acts of abstraction, reduce the complex idea which arises in the mind into others that are progressively simpler. *Body*, divested of its essential characters, presents the mere idea of *surface;* a surface, considered apart from its peculiar qualities, exhibits only *linear boundaries;* and a line, abstracting its continuity, leaves nothing in the imagination but the *points* which form its extremities. A solid is bounded by *surfaces;* a surface is circumscribed by *lines;* and a line is terminated by *points.* A point marks *position;* a line measures *distance;* and a surface represents *extension.* A line has only *length;* a surface has both *length* and *breadth;* and a solid combines all the three dimensions of *length, breadth,* and *thickness.*

The uniform description of a line which through its whole extent stretches in the same direction, gives the idea of a *straight* line. No more than one straight line can therefore join two points.

From our idea of the straight line is derived that of a *plane* surface, which, though more complex, has a

like uniformity of character. A straight line joining any two points situate in a plane, lies wholly on the surface; and consequently planes admit, in every way, a mutual and perfect application.

Two points ascertain the position of a straight line; for the line may continue to turn about one of the points till it falls upon the other. But to determine the position of a plane, it requires *three* points; because a plane touching the straight line which joins two of the points, may be made to revolve, till it meets the third point.

The separation or opening of two straight lines at their point of intersection, constitutes an *angle*. If we obtain the idea of *distance*, or linear extent, from *progressive* motion, we derive that of *divergence*, or angular magnitude, from *revolving* motion *.

GEOMETRY is divided into Plane and Solid; the former confining its views to the properties of space delineated on the same plane; the latter embracing the relations of different planes or surfaces, and of the solids which these describe or terminate. In the following definitions, therefore, the points and lines are all considered as existing in the same plane.

---

* See Note I.

BOOK I.

DEFINITIONS.

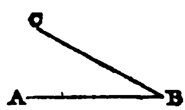
1. A *crooked* line is that which consists of straight lines not continued in the same direction.

2. A *curved* line is that of which no portion is a straight line.

3. The straight lines which contain an *angle* are termed its *sides*, and their point of origin or intersection, its *vertex*.

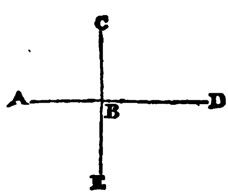
To abridge the reference, it is usual to denote an angle by tracing over its sides; the letter at the vertex, which is common to them both, being placed in the middle.

Thus, the angle contained by the straight lines AB and BC, or the opening formed by turning BA about the point B into the position BC, is named ABC or CBA,

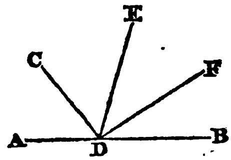


4. A *right angle* is the fourth part of an entire circuit or revolution.

If a straight line CB stand at equal angles CBA and CBD on another straight line AD, and if the surface ACD be laid over towards the opposite part, the point B and the line AD remaining the same; CB will, in this new position EB, make angles EBA and EBD equal to the former, and therefore all of them equal to each other. But the four angles ABC, CBD, DBE and EBA constitute about the point B a complete revolution; or the line BA in forming them, by its successive openings, would return into its original place,—and consequently each of those angles is a *right angle*.



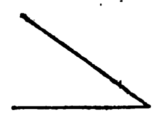
The angle contained by the opposite portions DA and DB of a straight line is hence equal to two right angles; and, for the same reason, all the angles ADC, CDE, EDF and FDB, formed at the point D and on the same side of the straight line AB, are together equal to two right angles.



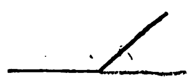
It is manifest that all right angles, being derived from the same measure, must be equal to each other.

5. The sides of a right angle are said to be *perpendicular* to each other.

6. An *acute* angle is less than a right angle.

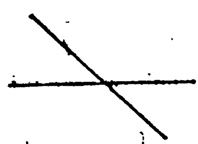


7. An *obtuse* angle is greater than a right angle.



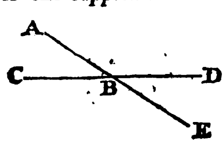
8. One side of an angle forms with the other produced a *supplemental* or *exterior angle.*

9. A *vertical* angle is formed by the production of both its sides.



10. The retroflected divergence of the two sides, or the defect of the angle from four right angles, is named a *reverse* angle.

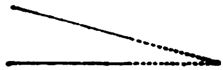
The angle DBE is *vertical* to ABC, ABD is the *supplemental* or *exterior* angle, and the angle made up of ABD, DBE, and EBC, or the opening formed by the regression of AB through the points D and E into the position BC, is the *reverse* angle.



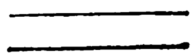
It is apparent that vertical angles, or those formed by the same

lines in opposite directions, must be equal; for the angles CBA and ABD which stand on the straight line CD, being equal to two right angles, are equal to ABD and DBE, and, omitting the common angle ABD, there remains CBA equal to DBE.

11. Two straight lines are said to be *inclined* to each other, if they meet when produced; and the angle so formed is called their *inclination*.



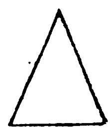
12. Straight lines which have no inclination, are termed *parallel*.



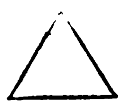
13. A *figure* is a plane surface included by a linear boundary called its *perimeter*.

14. Of rectilineal figures, the *triangle* is contained by *three* straight lines.

15. An *isosceles* triangle is that which has two of its sides equal.



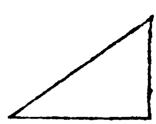
16. An *equilateral* triangle is that which has all its sides equal.



17. A triangle whose sides are unequal, is named *scalene*.

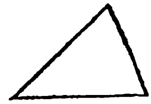
It will be shown (I. 10.) that every triangle has at least two acute angles. The third angle may therefore, by its character, serve to discriminate a triangle.

18. A *right-angled* triangle is that which has a right angle.



19. An *obtuse* angled triangle is that which has an obtuse angle.

20. An *acute* angled triangle is that which has *all* its angles acute.



21. Two triangles which are both of them right angled, or obtuse, or acute, are said to have the same ~~affection~~.
Character,

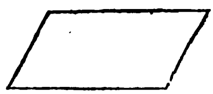
22. Any side of a triangle may be called its *base*, and the opposite angular point its *vertex*.

23. A *quadrilateral* figure is contained by *four* straight lines.

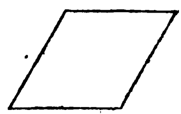
24. Of quadrilateral figures, a *trapezoid* (1) has two parallel sides:

25. A *trapezium* (2) has two of its sides parallel and the other two equal, though not parallel, to each other:

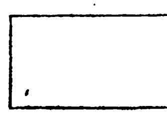
26. A *rhomboid* (3) has its opposite sides equal:



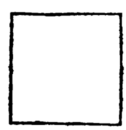
27. A *rhombus* (4) has all its sides equal:



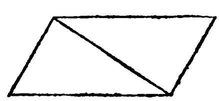
28. An *oblong*, or *rectangle*, (5) has a right angle, and its opposite sides equal:



29. A *square* (6) has a right angle, and all its sides equal.



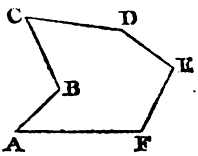
30. The straight line which joins obliquely the opposite angular points of a quadrilateral figure, is named a *diagonal*.



31. A rectilineal figure having more than four sides, bears the general name of a *polygon*.

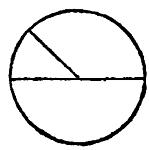
32. If an angle of a polygon be less than two right angles, it protrudes and is called *salient*; if it be greater than two right angles, it makes a sinuosity and is termed *re-entrant*.

Thus the angle ABC is re-entrant, and the rest of the angles of the polygon ABCDEF are salient at A, C, D, E and F.



33. A *circle* is a figure described by the revolution of a straight line about one of its extremities.

34. The fixed point is called the *centre* of the circle, the describing line its *radius*, and the boundary traced by the remote end of that line its *circumference*.



35. The *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

It is obvious that all radii of the same circle are equal to each other and to a semidiameter. It likewise appears from the slightest inspection, that a circle can only have one centre, and that circles are equal which have equal diameters.

36. Figures are said to be *equal*, when, applied to each other, they wholly coincide; they are *equivalent*, if, without coinciding, they yet contain the same space *.

* See Note II.

A PROPOSITION is a distinct portion of abstract science. It is either a *problem* or a *theorem*.

A PROBLEM proposes to effect some combination

A THEOREM advances some truth, which is to be established.

A *problem* requires *solution*, a *theorem* wants *demonstration*; the former implies an operation, and the latter generally needs a previous construction.

A *direct* demonstration proceeds from the premises, by a regular deduction.

An *indirect* demonstration attains its object, by showing that, any other hypothesis than the one advanced would involve a contradiction, or lead to an absurd conclusion.

A subordinate property, included in a demonstration, is sometimes, for the sake of unity, detached, and then it forms a LEMMA.

A COROLLARY is an obvious consequence that results from a proposition.

A SCHOLIUM is an excursive remark on the nature and application of a train of reasoning.

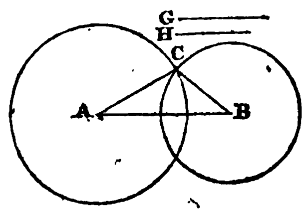
The operations in Geometry suppose the drawing of straight lines and the description of circles, or they require in practice the use of the rule and compasses.

PROPOSITION I. PROBLEM.

To construct a triangle, of which the three sides are given.

Let AB represent the base, and G, H two sides of the triangle, which it is required to construct.

From the centre A, with the distance G, describe a circle, and, from the centre B with the distance H, describe another circle, meeting the former in the point C: ACB is the triangle required.



Because all the radii of the same circle are equal, AC is equal to G; and, for the same reason, BC is equal to H. Consequently the triangle ACB answers the conditions of the problem.

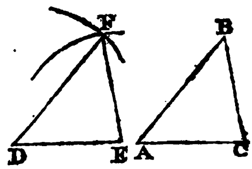
Corollary. If the radii G and H be equal to each other, the triangle will evidently be isosceles; and if those lines be likewise equal to the base AB, the triangle must be equilateral.—The limiting circles, after intersecting, must obviously diverge from each other, till, crossing the extension of the base AB, they return again and meet below it; thus marking two positions for the required triangle.

PROP. II. THEOREM.

Two triangles are equal, which have all the sides of the one equal to the corresponding sides of the other.

Let the two triangles ABC and DFE have the side AB equal to DF, AC to DE, and BC to FE: These triangles are equal.

For let the triangle ACB be applied to DEF, in the same position. The point A being laid on D, and the side AC on DE, their other extremities C and E must coincide, since AC is equal to DE. And because AB is equal to DF, the point B must be found in the circumference of a circle described from D with the distance DF; and, for the same reason, B must also be found in the circumference of a circle described from E with the distance EF: The vertex of the triangle ACB must, therefore, occur in a point which is common to both those circles, or, by the first proposition, in F the vertex of the triangle DFE. Consequently these two triangles, being rectilineal, must entirely coincide. The angle CAB is equal to EDF, ACB to DEF, and CBA to EFD; the equal angles being thus always opposite to the equal sides.

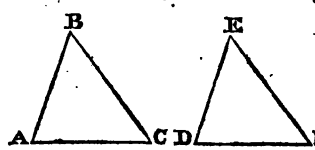


PROP. III. THEOR.

Two triangles are equal, if two sides and the angle contained by these in the one be respectively equal to two sides and the contained angle in the other.

Let ABC and DEF be two triangles, of which the side AB is equal to DE, the side BC to EF, and the angle ABC contained by the former equal to DEF which is contained by the latter: These triangles are equal.

For let the triangle ABC be applied to DEF: The vertex B being placed on E, and the side BA on ED, the extremity A must fall upon D, since AB is equal to DE. And because the angle or divergence ABC is equal to DEF, and the side AB coincides with DE, the other side BC must lie in the same direction with EF, and being of the same length, must entirely coincide with it; and consequently, the points A and C rest-



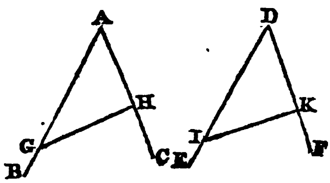
ing on D and F, the straight lines AC and DF will also coincide. Wherefore, the one triangle being thus perfectly adapted to the other, a general equality must obtain between them: The third sides AC and DF are hence equal, and the angles BAC, BCA opposite to BC and BA are equal respectively to EDF and EFD, which the corresponding sides EF and ED subtend.

PROP. IV. PROB.

At a point in a straight line, to make an angle equal to a given angle.

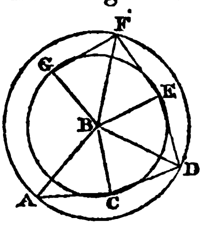
At the point D in the given straight line DE, to form an angle equal to the given angle BAC.

In the sides AB and AC of the given angle, assume the points G and H, join GH, from DE cut off DI equal to AG, and on DI constitute (I. 1.) a triangle DKI, having the sides DK and IK equal to AH and GH: EDK or EDF is the angle required.

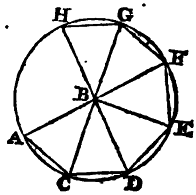


For all the sides of the triangles GAH and IDK being respectively equal, the angles opposite to the equal sides must be likewise equal (I. 2.), and consequently IDK is equal to GAH.

Scholium. If the segments AG, AH be taken equal, the construction will be rendered simpler and more commodious.—By the successive application of this problem, an angle may be continually multiplied. Two circles CEG and ADF being described from the vertex B of the given angle with radii BC and BA equal to its sides, and the base AC being repeated between those circumferences; a multitude of triangles are thus formed, all of them equal to the original triangle ABC. Consequently the angle ABD is double of ABC, ABE triple, ABF quadruple, ABG quintuple, &c.



If the sides AB and BC of the given angle be supposed equal, only one circle will be required, a series of equal isosceles triangles being constituted about its centre. It is evident that this addition is without limit, and that the angle so produced may continue to spread out, and its expanding side even make repeated revolutions.

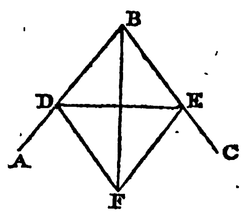


PROP. V. PROB.

To bisect a given angle.

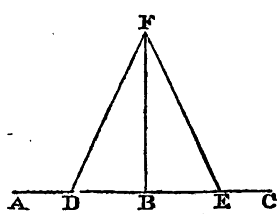
Let ABC be an angle which it is required to bisect.

In the side AB take any point D, and from BC cut off BE equal to BD; join DE, on which construct the isosceles triangle DFE (I. 1.), and draw the straight line BF: The angle ABC is bisected by BF.



For the two triangles DBF and EBF, having the side DB equal to EB, the side DF to EF, and BF common to both, are (I. 2.) equal, and consequently the angle DBF is equal to EBF.

Cor. Hence the mode of drawing a perpendicular from a given point B in the straight line AC; for the angle ABC, which the opposite segments BA and BC make with each other, being equal to two right angles, the straight line that bisects it must be the perpendicular required. Taking BD, therefore, equal to BE, and constructing the isosceles triangle DFE; the straight line BF which joins the vertex of the triangle, is perpendicular to AC.

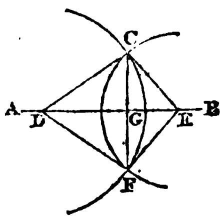


PROP. VI. PROB.

To let fall a perpendicular upon a straight line, from a given point without it.

From the point C, to let fall a perpendicular upon a given straight line AB.

In AB take the point D, and with the distance DC describe a circle; and in the same line take another point E, and with distance EC describe a second circle intersecting the former in F; join CF, crossing the given line in G: CG is perpendicular to AB.

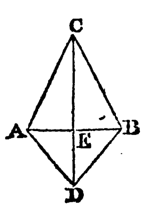


For the triangles DCE and DFE have the side DC equal to DF, EC to EF, and DE common to them both; whence (I. 2.) the angle CDE or CDG is equal to FDE or FDG. And because in the triangles DCG and DFG, the side DC is equal to DF, DG common, and the contained angles CDG and FDG are proved to be equal; these triangles are (I. 3.) equal, and consequently the angle DGC is equal to DGF, and each of them a right angle, or CG is perpendicular to AB.

PROP. VII. PROB.

To bisect a given finite straight line.

On the given straight line AB, construct two isosceles triangles (I. 1.) ACB and ADB, and join their vertices C and D by a straight line cutting AB in the point E: AB is bisected in E.

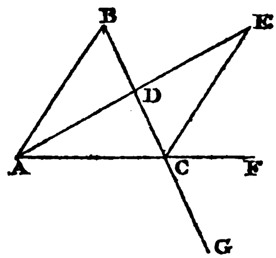


For the sides AC and AD of the triangle CAD being respectively equal to BC and BD of the triangle CBD, and the side CD common to them both; these triangles (I. 2.) are equal, and the angle ACD or ACE is equal to BCD or BCE. Again, the triangles ACE and BCE, having the side AC equal to BC, CE common, and the contained angle ACE equal to BCE, are (I. 3.) equal, and consequently the base AE is equal to BE.

PROP. VIII. THEOR.

The exterior angle of a triangle is greater than either of its interior opposite angles.

The exterior angle BCF, formed by producing a side AC of the triangle ABC, is greater than either of the opposite and interior angles CAB and CBA.



For bisect the side BC in D (I. 7.), draw AD, and produce it until DE be equal to AD, and join EC.

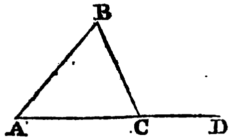
The triangles ADB and EDC have, by construction, the side DA equal to DE, the side DB to DC, and the vertical angle BDA equal to CDE; these triangles are, therefore, equal (I. 3.), and the angle DCE is equal to DBA. But the angle BCF is evidently greater than DCE; it is consequently greater than DBA or CBA.

In like manner, it may be shown, that if BC be produced, the exterior angle ACG is greater than CAB. But ACG is equal to the vertical angle BCF, and hence BCF must be greater than either the angle CBA or CAB.

PROP. IX. THEOR.

Any two angles of a triangle are together less than two right angles.

The two angles BAC and BCA of the triangle ABC are together less than two right angles.

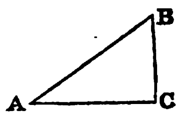


For produce the common side AC. And, by the last proposition, the exterior angle BCD is greater than BAC, add BCA to each, and the two angles BCD and BCA are greater than BAC and BCA, or BAC and BCA are together less than BCD and BCA, that is, less than two right angles (Def. 4).

PROP. X. THEOR.

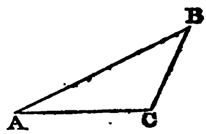
Every triangle has two acute angles.

The triangle ABC obviously may have one angle right, or obtuse or acute. And first, let it be right angled at C. By the last proposition, the angles ACB and CAB are less than two right angles, and so are the angles ACB and ABC.



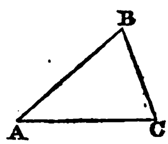
Consequently the angles CAB and CBA are each of them less than one right angle, or they are both acute.

Next, let the triangle have an obtuse angle ACB. The angles ACB and CAB, being less than two right angles, and ACB being greater than one right angle, CAB must be still less than a right angle. And the angles ACB and ABC being less than two right angles, ABC must also be



still less than one right angle. Consequently the angles CAB and CBA are both of them acute.

Lastly, let the triangle have the angle at C acute. If one of the remaining angles, such as BAC, be likewise acute, the two angles ACB and BAC are both of them acute. But if the angle BAC be either obtuse or a right angle, it



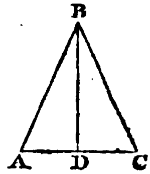
comes under the two former cases, and the other angles ABC and ACB are, therefore, acute.

PROP. XI. THEOR.

The angles at the base of an isosceles triangle are equal.

The angles BAC and BCA at the base of the isosceles triangle ABC are equal.

For draw (I. 5.) BD bisecting the vertical angle ABC.



Because AB is equal to BC, the side BD common to the two triangles BDA and BDC, and the angles ABD and CBD contained by them are equal; these triangles are equal (I. 3.) and consequently the angle BAD is equal to BCD.

Cor. Every equilateral triangle is also equiangular *.

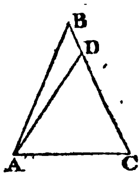
PROP. XII. THEOR.

If two angles of a triangle be equal, the sides opposite to them are likewise equal.

Let the triangle ABC have two equal angles BCA and BAC; the opposite sides AB and BC are also equal.

For if AB be not equal to CB, let it be equal to CD, and join AD.

Comparing now the triangles BAC and DCA, the side AB is by supposition equal to CD, AC is common to both, and the contained angle BAC is equal to DCA; the two triangles (I. 3.) are, therefore, equal. But this conclusion is manifestly absurd. To suppose then the inequality of AB and BC involves a contradiction; and consequently those sides must be equal.



Cor. Every equiangular triangle is also equilateral.

PROP. XIII. THEOR.

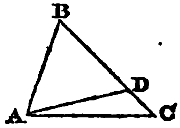
In a triangle, that angle is the greater which lies opposite to a greater side.

If a side BC of the triangle ABC be greater than BA; the opposite angle BAC is greater than BCA.

---

* *See Note III.*

For make BD equal to BA, and join AD. The angle CAB is greater than DAB; but since BA is equal to BD, the angle DAB (I. 11.) is equal to ADB, and consequently CAB is greater than ADB. Again, the angle ADB, being an exterior angle of the triangle CAD, is (I. 8.) greater than ACD or ACB; wherefore the angle CAB is much greater than ACB.

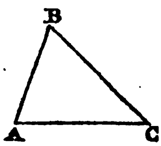


PROP. XIV. THEOR.

That side of a triangle is the greater which subtends a greater angle.

If, in the triangle ABC, the angle CAB be greater than ACB; its opposite side BC is greater than AB.

For if BC be not greater than AB, it must be either equal or less. But it cannot be equal, because the angle CAB would then be equal to ACB (I. 11.); nor can BC be less than AB, for then AB would be greater than BC, and consequently (I. 13.) the angle ACB would be greater than CAB, or CAB less than ACB, which is absurd. The side BC being thus neither equal to AB, nor less than it, must therefore be greater than AB.



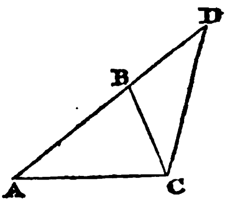
PROP. XV. THEOR.

Two sides of a triangle are together greater than the third side.

The two sides AB and BC of the triangle ABC are together greater than the third side AC.

For produce AB until DB be equal to the side BC, and join CD.

Because BC is equal to BD, the angle BCD is equal to BDC (I. 11.); but the angle ACD is greater than BCD, and therefore greater than BDC, or ADC; consequently the opposite side AD is greater than AC (I. 14.); and since AD is equal to AB and BD, or to AB and BC, the two sides AB and BC are together greater than the third AC *.

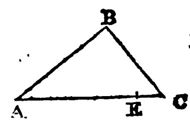


Cor. By an extension of this proposition, it may be shown that a straight line is the shortest line which will connect two points †.

PROP. XVI. THEOR.

The difference between two sides of a triangle is less than the third side.

Let the side AC be greater than AB, and from it cut off a part AE equal to AB; the remainder EC is less than the third side BC.



For the two sides AB and BC are together greater than AC (I. 15.); take away the equal lines AB and AE, and there remains BC greater than EC, or EC is less than BC ‡.

PROP. XVII. THEOR.

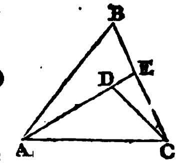
Two straight lines drawn to a point within a triangle from the extremities of its base, are together less than the sides of the triangle, but contain a greater angle.

The straight lines AD and CD, projected to a point D within the triangle ABC from the extremities of the base AC, are together less than the sides AB and CB of the triangle, but contain a greater angle.

For produce AD to meet CB in E. The two sides AB and BE of the triangle ABE are greater than the third side AE (I. 15.); add EC to each, and AB, BE, EC, or AB and BC,

* *See Note IV.* † See Note V. ‡ *See Note VI.*

are greater than AE and EC. But the sides CE and ED of the triangle DEC are (I. 15.) greater than DC, and consequently CE, ED, together with DA, or CE and EA, are greater than CD and DA. Wherefore the sides AB and BC, being greater than AE and EC, which are themselves greater than AD and DC, must be still greater than AD and DC, or the lines AD and DC are less than AB and BC, the sides of the triangle.



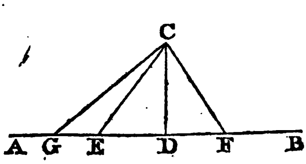
Again, the angle ADC, being the exterior angle of the triangle DCE, is greater than DEC (I. 8.); and, for the same reason, DEC is greater than ABE, the opposite interior angle of the triangle EAB. Consequently ADC is still greater than ABE or ABC.

PROP. XVIII. THEOR.

If straight lines be drawn from the same point to another straight line, the perpendicular is the shortest of them all; the lines equidistant from it on both sides are equal; and those more remote are greater than such as are nearer.

Of the straight lines CG, CE, CD, and CF drawn from a given point C to the straight line AB, the perpendicular CD is the least, the equidistant lines CE and CF are equal, but the remoter line CG is greater than either of these two.

For the right angle CDE, which is equal to CDF, is (I. 8.) greater than the interior angle CFD of the triangle DCF, and consequently the opposite side CF is (I. 14.) greater than CD, or CD is less than CF.



But a straight line drawn of a determinate length from C to AB, may have two positions; for, if CE be supposed to turn about the point C, the angle CEA will (I. 8.) continually decrease, till, passing from obtuse to acute, it becomes equal to CEF,

and then forms (I. 12.) the isosceles triangle ECF.—Because ED then is by hypothesis equal to FD, CD common to the two triangles ECD and FCD, and the contained angles CDE and CDF equal; these triangles (I. 3.) are equal, and consequently their bases CE and CF are equal.

Again, because GCD is a right angled triangle, the angle CGD or CGE is acute (I. 10.), and, for the same reason, the angle CED of the triangle CDE is acute, and consequently its adjacent angle CEG is obtuse. Wherefore CEG is still greater than CGE, and the opposite side CG greater (I. 14.) than CE.

Cor. Hence only a single perpendicular CD can be let fall from the same point C upon a given straight line AB; and hence also a pair only of equal straight lines greater than CD can at once be extended from C to AB, making on the same side, the one an obtuse angle CEA, and the other an acute angle CFA.—As the term *distance* signifies the shortest road, the distance between two points is the straight line which joins them; and the distance from a point to a straight line, is the perpendicular let fall upon it.

PROP. XIX. THEOR.

If two sides of one triangle be respectively equal to those of another, but contain a greater angle; the base also of the former will be greater than that of the latter.

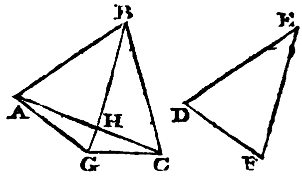
In the triangles ABC and DEF, let the sides AB and BC be equal to DE and EF, but the angle ABC greater than DEF; then is the base AC greater than DF.

For, suppose AB one of the sides to be not greater than BC or EF, and (I. 4.) draw BG equal to EF making an angle ABG equal to DEF, join AG and GC.

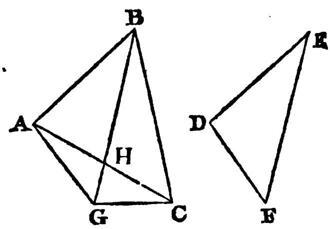
Because AB and BG are equal to DE and EF, and the

contained angle ABG is equal to DEF; the triangles ABG and DEF (I. 3.) are equal, and have equal bases AG and DF.

First, let the triangles, ABC and DEF be isosceles. Since the side AB is equal to BC, the angle BAC (I. 11.) is equal to BCA; but (I. 8.), the angle BHC is greater than BAH or BCH, and consequently (I. 14.) the side BC or BG is greater than BH, or the point G lies beyond H.



Next, suppose the side BC or EF to be greater than AB or DE. Wherefore (I. 13.) the angle BAC is greater than BCA; but (I. 8.) the exterior angle BHC of the triangle ABH is greater than BAH or BAC, and hence still greater than BCA or BCH; consequently the side BC or EF is (I. 14.) greater than BH.



In every case, therefore, the point G must lie below the base AC. But the triangle GBC being evidently isosceles, its angles BGC and BCG (I. 11.) are equal. Whence the angle AGC, being greater than BGC or BCG, which again is greater than ACG, must be still greater than ACG; and therefore the opposite side AC is (I. 14.) greater than AG or DF *.

PROP. XX. THEOR.

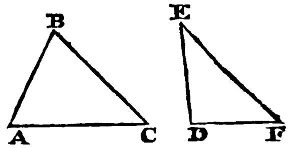
If two sides of one triangle be respectively equal to those of another, but stand on a greater base; the angle contained by the former will be likewise greater than what is contained by the latter.

Let the triangles ABC and DEF have the sides AB and BC equal to DE and EF, but the base AC greater than DF; the vertical angle ABC is greater than DEF.

---

* *See Note VII.*

For if ABC be not greater than the angle DEF, it must either be equal or less. But it cannot be equal to DEF, for the sides AB, BC being then equal to DE, EF, and containing equal angles, the base AC would (I. 3.) be equal to DF, which is contrary to the hypothesis. Still more absurd it would be to suppose the angle ABC less than DEF, since the triangles BAC and EDF, having their sides AB, BC equal to DE, EF, but the contained angle ABC less than DEF, or DEF greater than ABC, the base DF would, from the preceding proposition, be greater than AC, or AC would be less than DF *.

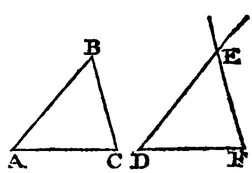


PROP. XXI. THEOR.

Two triangles are equal, which have two angles and a corresponding side in the one respectively equal to those in the other.

Let the triangles ABC and DEF have the angle BAC equal to EDF, the angle BCA to EFD, and a side of the one equal to a side of the other, whether it be interjacent or opposite to those equal angles; the triangles will be equal.

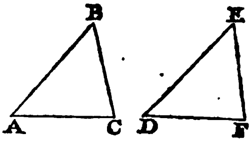
First, let the equal sides be AC and DF, which are interjacent to the equal angles in both triangles.—Apply the triangle ABC to DEF; the point A being laid on D, and the straight line AC on DF, the other extremities C and F must coincide, since those lines are equal. And because the angle BAC is equal to EDF, and the side AC is applied to DF, the other side AB must lie along DE; and for the same reason, the angles BCA and EFD being equal, the side CB must lie along FE. Wherefore the point B, which is common to both the lines AB and CB, will be found likewise in both DE and FE; that is, it



* See Note VIII.

must fall upon the corresponding vertex E. The two triangles ABC and DEF, thus adapting, are hence entirely equal.

Next, let the equal sides be AB and DE, which are opposite to the equal angles BCA and EFD. The triangle ABC being laid on DEF, the sides AB and AC of the angle BAC will apply to DE and DF, the sides of the equal angle EDF; and since AB is equal to DE, the points B and E must coincide; but, by hypothesis, the angles BCA and EFD being equal, BC must adapt itself to EF, for otherwise one of those angles becoming exterior, would (I. 8.) be greater than the other. Whence the triangles ABC, DEF are entirely coincident, and have those sides equal which subtend equal angles.

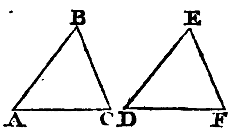


PROP. XXII. THEOR.

Two triangles are equal, which, being of the same affection, have two sides and an opposite angle in the one equal to those in the other.

Let the triangles ABC and DEF have the side AB equal to DE, BC to EF, and the angles BAC, EDF, opposite to BC, EF, also equal; the triangles themselves are equal, if both the angles BCA and EFD be right, or acute, or obtuse.

For, the triangle ABC being applied to DEF, the angle BAC will adapt itself to EDF, since they are equal; and the point B must coincide with E, because the side AB is equal to DE. But the other equal sides BC and EF, now stretching from the same point E towards DF, must likewise coincide; for if the angle at C or F be right, there can exist no more than one perpendicular EF (I. 18. cor.) and, in like manner, if this angle at F be either obtuse or acute, the line EF, which forms it, can have only one corresponding po-



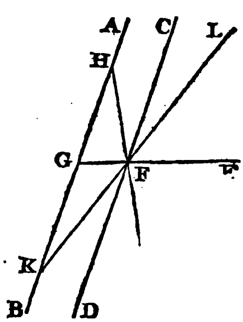
sition.—Whence, in each of these three cases, the triangle ABC admits of a perfect adaptation with DEF *.

PROP. XXIII. THEOR.

If a straight line fall upon two parallel straight lines, it will make the alternate angles equal, the exterior angle equal to the interior opposite one, and the two interior angles on the same side together equal to two right angles.

Let the straight line EFG fall upon the parallels AB and CD; the alternate angles AGF and DFG are equal, the exterior angle EFC is equal to the interior angle EGA, and the interior angles CFG and AGF, or FGB and GFD, are together equal to two right angles.

For suppose the straight line EFG, produced both ways from F, to turn about that point in the direction BA; it will first cut the extended line AB towards A, and will in its progress afterwards meet the same line on the other side towards B. In the position IFH, the angle EFH is the exterior angle of the triangle FHG, and therefore greater than FGH or EGA (I. 8.) But in the last position LFK, the exterior angle EFL is equal to its vertical angle GFK in the triangle FKG, and to which the angle FGA is exterior; consequently (I. 8.) FGA is greater than EFL, or the angle EFL is less than FGA or EGA. When the incident line EFG, therefore, meets AB above the point G, it makes an angle EFH *greater* than EGA; and when it meets AB below that point, it makes an angle EFL, which is *less* than the same angle. But in passing through all the degr



* See Note IX.

greater to less, a varying magnitude must evidently rencounter, as it proceeds, the single intermediate limit of equality. Wherefore, there is a certain position, CD, in which the line revolving about the point F makes the exterior angle EFC *equal* to the interior EGA, and at the same time meets AB neither towards the one part nor the other, or is parallel to it.

And now, since EFC is proved to be equal to EGA, and is also equal to the vertical angle GFD; the alternate angles FGA and GFD are equal. Again, because GFD and FGA are equal, add the angle FGB to each, and the two angles GFD and FGB are equal to FGA and FGB; but the angles FGA and FGB, on the same side of AB, are equal to two right angles, and consequently the interior angles GFD and FGB are likewise equal to two right angles.

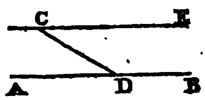
Cor. Since the position CD is individual, or that only *one* straight line can be drawn through the point F parallel to AB, it follows that the converse of the proposition is likewise true, and that those three properties of parallel lines are *criteria* for distinguishing parallels *.

PROP. XXIV. PROB.

Through a given point, to draw a straight line parallel to a given straight line.

To draw, through the point C, a straight line parallel to AB.

In AB take any point D, join CD, and at the point C make (I. 4.) an angle DCE equal to CDA; CE is parallel to AB.



For the angles CDA and DCE, thus formed equal, are the alternate angles which CD makes with the straight lines CE and AB, and, therefore, by the corollary to the last proposition, these lines are parallel.

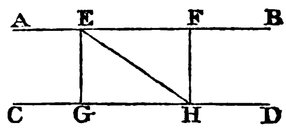
* See Note X.

PROP. XXV. THEOR.

Parallel lines are equidistant, and equidistant straight lines are parallel.

The perpendiculars EG, FH, let fall from any points E, F in the straight line AB upon its parallel CD, are equal; and if these perpendiculars be equal, the straight lines AB and CD are parallel.

For join EH: and because each of the interior angles EGH and FHG is a right angle, they are together equal to two right angles, and consequently the perpendiculars EG and FH are (I. 23. cor.) parallel to each other; wherefore (I. 23.) the alternate angles HEG and EHF are equal. But, EF being parallel to GH, the alternate angles EHG and HEF are likewise equal; and thus the two triangles HGE and HFE, having the angles HEG and EHG respectively equal to EHF and HEF, and the side EH common to both, are (I. 21.) equal, and hence the side EG is equal to FH.



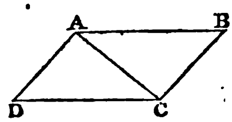
Again, if the perpendiculars EG and FH be equal, the two triangles EGH and EFH, having the side EG equal to FH, EH common, and the contained angle HEG equal to EHF, are (I. 3.) equal, and therefore the angle EHG equal to HEF, and (I. 23.) the straight line AB parallel to CD.

PROP. XXVI. THEOR.

The opposite sides of a rhomboid are parallel.

If the opposite sides AB, DC, and AD, BC of the quadrilateral figure ABCD be equal, they are also parallel.

For join AC. And because AB is equal to DC, BC to AD, and AC is common; the two triangles ABC and ADC are (I. 2.) equal. Consequently the angle ACD is equal to CAB, and the side AB (I. 23. cor.) parallel to CD; and, for the same reason, the angle CAD is equal to ACB, whence the side AD is parallel to BC.



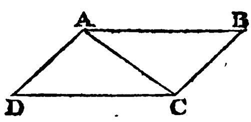
Cor. Hence the angles of a square or rectangle are all of them right angles; for the opposite sides being equal, are parallel; and if the angle at A be right, the other interior one at B is also a right angle (I. 25.), and consequently the angles at C and D, opposite to these, are right.

PROP. XXVII. THEOR.

The opposite sides and angles of a parallelogram are equal.

Let the quadrilateral figure ABCD have the sides AB, BC parallel to CD, AD; these are respectively equal, and so are the opposite angles at A and C, and at B and D.

For join AC. Because AB is parallel to CD, the alternate angles BAC and ACD are (I. 25.) equal; and since AD is parallel to BC, the alternate angles ACB and CAD are likewise equal. Wherefore the triangles ABC and ADC, having the angles CAB and ACB equal to ACD and CAD, and the interjacent side AC common to both, are (I. 21.) equal. Consequently, the side AB is equal to CD, and the side BC to AD; and these opposite sides being thus equal, the opposite angles (I. 26.) must also be equal.



Cor. Hence the diagonal divides a rhomboid or parallelogram into two equal triangles. Hence also an oblong is a rectangular parallelogram; for if the angle at A be right, the

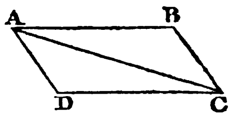
opposite angle at C is right, and the remaining angles at B and D, being equal to each other and to two right angles, must be right angled.

PROP. XXVIII. THEOR.

If the parallel sides of a trapezoid be equal, the other sides are likewise equal and parallel.

Let the sides AB and DC be equal and parallel; the sides AD and BC are themselves equal and parallel.

For join AC. Because AB is parallel to CD, the alternate angles CAB and ACD are (I. 23.) equal; and the triangles ABC and ADC, having the side AB equal to CD, AC common to both, and the contained angle CAB equal to ACD, are, therefore, equal (I. 3.) Whence the side BC is equal to AD, and the angle ACB equal to CAD; but these angles being alternate, BC must also be parallel to AD (I. 23. cor.)

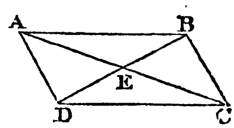


PROP. XXIX. THEOR.

The diagonals of a rhomboid mutually bisect each other.

If the diagonals of the rhomboid ABCD intersect each other in E; the part AE is equal to CE, and DE to BE.

For because a rhomboid is also a parallelogram (I. 26.), the alternate angles BAC and ACD are equal (I. 23.) and likewise ABD and BDC. The triangles AEB and CED, having thus the angles BAE, ABE respectively equal to DCE and CDE, and the interjacent sides AB and CD equal, are (I. 21.) wholly equal. Wherefore AE is equal to the corresponding side CE, and BE to DE.

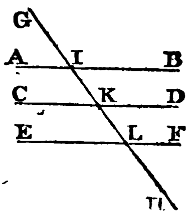


Cor. Hence the diagonals of a rectangle are equal to each other; for if the angles at A and B were right angles, the triangles DAB and CBA would be equal (I. 3.) and consequently the base DB equal to AC.

PROP. XXX. THEOR.

Lines parallel to the same straight line, are parallel to each other.

If the straight line AB be parallel to CD, and CD parallel to EF; then is AB parallel to EF.



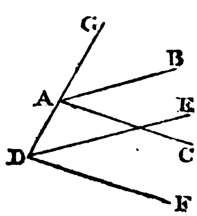
For let a straight line GH cut these lines.

And because AB is parallel to CD, the exterior angle GIA is equal (I. 25.) to the interior GKC; and since CD is parallel to EF, this angle GKC is, for the same reason, equal to GLE. Therefore the angle GIA is equal to GLE, and consequently AB is parallel to EF (I. 23. cor.)

PROP. XXXI. THEOR.

Straight lines drawn parallel to the sides of an angle, contain an equal angle.

If the straight lines AB, AC be parallel to DE, DF; the angle BAC is equal to EDF.



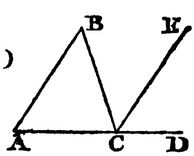
For draw the straight line GAD through the vertices. And since AC is parallel to DF, the exterior angle GAC is (I. 23.) equal to GDF; and, for the same reason, GAB is equal to GDE; there consequently remains the angle BAC equal to EDF.

PROP. XXXII. THEOR.

An exterior angle of a triangle is equal to both its opposite interior angles, and all the interior angles of a triangle are together equal to two right angles.

The exterior angle BCD, formed by the production of the side AC of the triangle ABC, is equal to the two opposite interior angles CAB and CBA, and all the interior angles CAB, CBA and BCA of the triangle are together equal to two right angles.

For, through the point C, draw (I. 24.) the straight line CE parallel to AB. And, AB being parallel to CE, the interior angle BAC is (I. 23.) equal to the exterior one ECD; and, for the same reason, the alternate angle ABC is equal to BCE. Wherefore the two angles CAB and ABC are equal to DCE and ECB, or to the whole exterior angle BCD. Add to each the adjacent angle BCA; and all the interior angles of the triangle ABC are together equal to the angles BCD and BCA on the same side of the straight line AD, that is, to two right angles.



Cor. 1. Hence the two acute angles of a right angled triangle are together equal to one right angle; and hence each angle of an equilateral triangle is two third parts of a right angle.

Cor. 2. Hence if a triangle have its exterior angle, and one of its opposite interior angles, double of those in another triangle; its remaining opposite interior angle will also be double of the corresponding angle in the other *.

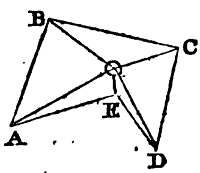
---

* *See Note XI.*

PROP. XXXIII. THEOR.

The angles round any rectilineal figure are together equal to twice as many right angles (abating four from the result) as the figure has sides.

For assume a point O within the figure, and draw straight lines OA, OB, OC, OD, and OE, to the several corners. It is obvious, that the figure is thus resolved into as many triangles as it has sides, and whose collected angles must be therefore equal to twice as many right angles. But the angles at the bases of these triangles constitute the internal angles of the figure. Consequently, from the whole amount there is to be deducted the vertical angles about the point O, and which are (Def. 4.) equal to four right angles.



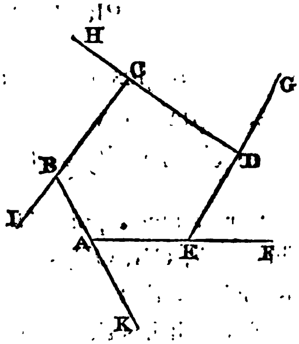
Cor. Hence all the angles of a quadrilateral figure are equal to four right angles, those of a pentelateral figure equal to six right angles, and so forth; increasing the amount by two right angles, for each additional side.

PROP. XXXIV. THEOR.

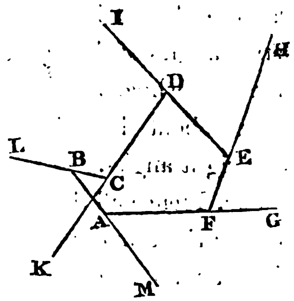
The exterior angles of a rectilineal figure are together equal to four right angles.

The exterior angles DEF, CDG, BCH, ABI, and EAK of the rectilineal figure ABCDE are taken together equal to four right angles.

For each exterior angle DEF, with its adjacent interior one AED, is equal to two right angles. All the exterior angles, therefore, added to the interior angles, are equal to twice as many right angles as the figure has sides. Consequently the exterior angles are equal to the four right angles which, by the last Proposition, were abated, to form the aggregate of the interior angles.



Cor. If the figure has a re-entrant angle BCD, the angle BCK which occurs in place of an exterior angle, must be taken away in forming the amount; for the corresponding interior angle BCD, in this case, exceeds two right angles, by BCK. Hence the angles EFG, DEH, CDI, ABL, FAM, diminished by BCK, are equal to four right angles.

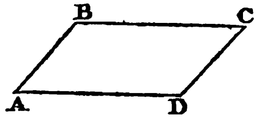


PROP. XXXV. THEOR.

If the opposite angles of a quadrilateral figure be equal, its opposite sides will be likewise equal and parallel.

In the quadrilateral figure ABCD, let the angle at B be equal to the opposite one at D, and the angle at A equal to that at C; the sides AB and BC are equal and parallel to DC and DA.

For all the angles of the figure being equal to four right angles (I. 33. cor.), and the opposite angles being mutually equal, each pair of adjacent angles must be equal to two right angles. Wherefore ABC and BCD are equal to two right angles, and the lines AB and DC (I. 23. cor.) parallel; for the same reason, ABC and BAD being together equal to two right angles, the sides BC and AD, which limit them, are parallel. But (I. 27.) the parallel sides of the figure are also equal.



Cor. Hence a rectangle has its opposite sides equal and parallel.

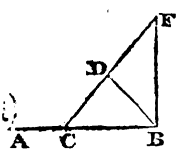
PROP. XXXVI. PROB.

To draw a perpendicular from the extremity of a given straight line.

From the point B, to draw a perpendicular to AB, without producing that line.

In AB take any point C, and on BC (I. 1. cor.) describe an isosceles triangle BDC, produce CD till DF equal it; and BF being joined, is the perpendicular required.

For, since by construction DF is equal to CD or BD, the triangle BDF is isosceles, and (I. 11.) the angle DBF equal to DFB; whence the angle CDB, being equal (I. [illegible].) to the interior angles DBF and DFB, is double of DBF, or the angle DBF is half of CDB. But the triangle BDC being isosceles, the angle CBD is equal to BCD; consequently the angles DBF and DBC are the halves of the vertical and base angles of BDC, and therefore (I. 32.) the whole angle CBF is the half of two right angles, or it is equal to one right angle *.



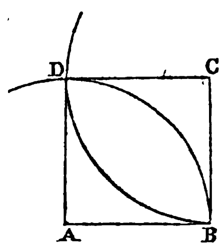
* See Note XII.

PROP. XXXVII. PROB.

On a given finite straight line, to construct a square.

Let AB be the side of the square which it is required to construct.

From the extremity B draw (I. 36.) BC perpendicular to BA and equal to it, and, from the points A and C with the distance BA or BC describe two circles intersecting each other in the point D, join AD and CD; the quadrilateral figure ABCD is the square required.



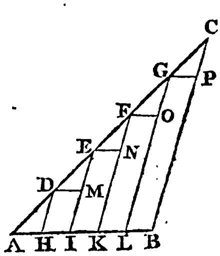
For, by this construction, the figure has all its sides equal, and one of its angles ABC a right angle; which comprehends the whole of the definition of a square.

PROP. XXXVIII. PROB.

To divide a given straight line into any number of equal parts.

Let it be required to divide the straight line AB into a given number of equal parts, suppose five.

From the point A and at any oblique angle with AB, draw a straight line AC, in which take the portion AD, and repeat it five times from A to C, join CB, and from the several points of section D, E, F, and G draw the parallels DH, EI, FK, and GL, (I. 24.), cutting AB in H, I, K, and L: AB is divided at these points into five equal parts.



For (I. 24.) draw DM, EN, FO, and GP parallel to AB. And because DH is parallel to EM, the exterior angle ADH is equal to DEM (I. 23.); and, for the same reason, since AH is parallel to DM, the angle DAH is equal to EDM. Wherefore the triangles ADH and DEM, having two angles respectively equal and the interjacent sides AD, DE—are (I. 21.) equal, and consequently AH is equal to DM. In the same manner, the triangle ADH is proved to be equal to EFN, to FGO, and GCP, and therefore their bases EN, FO, and GP are all equal to AH. But these lines are equal to HI, IK, KL, and LB, for the opposite sides of parallelograms are equal (I. 29.). Wherefore the several segments AH, HI, IK, KL, and LB, into which the straight line AB is divided, are all equal to each other.

Scholium. The construction of this problem may be facilitated in practice, by drawing from B in the opposite direction a straight line parallel to AC, and repeating on both of them portions equal to the assumed segment AD, but only four times, or one fewer than the number of divisions required; then joining D, the first section of AC, with the last of its parallel, E with the next, and so on till G, which connecting lines are (I. 28.) all parallel, and consequently the former demonstration still holds.

ELEMENTS

OF

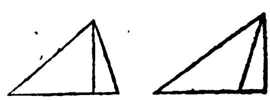
GEOMETRY.

BOOK II.

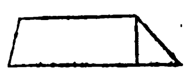
DEFINITIONS.

1. In a right-angled triangle, the side that subtends the right angle is termed the *hypotenuse*; either of the sides which contain it, the *base*; and the other side, the *perpendicular*.

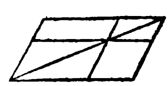
2. The *altitude* of a triangle is a perpendicular let fall from the vertex upon the base or its extension.



3. The *altitude* of a trapezoid is the perpendicular drawn from one of its parallel sides to the other.



4. The *complements* of rhomboids about the diagonal of a rhomboid, are the spaces required to complete the rhomboid; and the defect of each rhomboid from the whole figure, is termed a *gnomon*.



5. A rhomboid or rectangle is said to be *contained* by any two adjacent sides.

A rhomboid is often indicated merely by the two letters placed at opposite corners.

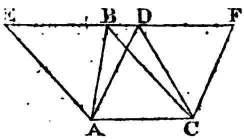
PROP. I. THEOR.

Triangles, which have the same altitude, and stand on the same base, are equivalent.

The triangles ABC and ADC which stand on the same base AC and have the same altitude, contain equal spaces.

For, join the vertices B, D by a straight line, which produce both ways; and from A draw AE (I. 24.) parallel to CB, and from C draw CF parallel to AD.

Because the triangles ABC, ADC have the same altitude, the straight line EF is parallel to AC (I. 25.), and consequently the figures CE and AF are parallelograms. Wherefore EB, being equal to AC (I. 27.), which is equal to DF, is itself equal to DF. Add BD to each,



and ED is equal to BF; but EA is equal to BC (I. 27.), and the interior angle AED is equal to the exterior angle CBF (I. 23.). Thus the two triangles EDA, BFC have the sides ED, EA equal to BF, BC, and the contained angle AED equal to CBF, and are therefore equal (I. 3.). Take these equal triangles CBF and EDA from the whole quadrilateral space AEFC, and there remains the rhomboid AEBC equivalent to ADFC. Whence the triangles ABC and ADC, which are the halves of these rhomboids (I. 27. cor.), are likewise equivalent.

Cor. Hence the rhomboids on the same base and between the same parallels, are equivalent.

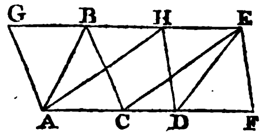
PROP. II. THEOR.

Triangles which have the same altitude and stand on equal bases, are equivalent.

The triangles ABC, DEF, standing on equal bases AC and DF and having the same altitude, contain equal spaces.

For let the bases AC, DF be placed in the same straight line, join BE, and produce it both ways, draw AG and DH parallel to CB and FE (I. 24.), and join AH, CE.

Because the triangles ABC, DEF are of equal altitude, GE is parallel to AF (I. 25.), and GC, HF are parallelograms. But AC, being equal to DF, and DF equal (I. 27.) to HE, it must also be equal to HE, and therefore (I. 28.) AE is a rhomboid or parallelogram.



Whence the rhomboid GC is equivalent to AE (II. 1. cor.), and this again is, for the same reason, equivalent to HF; consequently GC is ~~equal~~ equiv. to HF, and therefore their halves or (I. 27. cor.) the triangles ABC and DEF are equivalent.

Cor. Hence rhomboids on equal bases and between the same parallels, are equivalent.

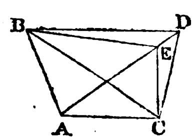
PROP. III. THEOR.

Equivalent triangles on the same or equal bases, have the same altitude.

If the triangles ABC and ADC, standing on the same base AC, contain equal spaces, they have the same altitude, or the straight line which joins their vertices is parallel to AC.

For if BD be not parallel to AC, draw the parallel BE meeting AD or that side produced, in E, and join CE.

Because BE is made parallel to AC, the triangle ABC is (II. 1.) equivalent to AEC; but ABC is by hypothesis equivalent to ADC, and therefore AEC is equivalent to ADC, which is absurd. The supposition then that BD is not parallel to AC, involves a contradiction.



The same mode of demonstration, it is obvious, will apply in the case where the equivalent triangles stand on equal bases.

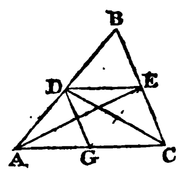
Cor. Hence equivalent rhomboids on the same or equal bases, have the same altitude.

PROP. IV. THEOR.

A straight line bisecting two sides of a triangle, is parallel to the base.

The straight line DE which joins the middle points of the sides AB and BC, is parallel to the base AC of the triangle ABC.

For join AE and CD. Because the triangles ADC, BCD stand on equal bases AD, DB, and have the same vertex or altitude, they are (II. 2.) equivalent, and therefore ADC is half of the whole triangle ABC. For the same reason, since CE is equal to EB, the triangle AEC is equivalent to AEB, and is consequently half of the whole triangle ABC. Whence the triangles ADC and AEC are equivalent; and they stand on the same base AC, and have therefore the same altitude (II. 3.), or DE is parallel to AC.



Cor. Hence the triangle DBE cut off by the line DE, is the fourth part of the original triangle. For bisect AC in G, and join DG, which is therefore parallel to BC. The triangle ADG is equivalent to GDC (II. 2.), and GDC, being the half of the rhomboid GE, is equivalent to DEC, which again is (II. 2.) equivalent to DEB. The triangle ABC is thus divided into four equivalent triangles, of which DBE is one. Hence also the rhomboid GDEC is half of the original triangle*.

PROP. V. PROB.

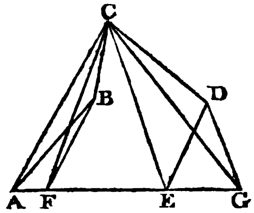
To find a triangle equivalent to any rectilineal figure.

---

* *See Note XIII.*

Let it be required to reduce the five-sided figure ABCDE to a triangle, or to find a triangle that shall contain an equal space.

Join any two alternate points A, C, and, through the intermediate point B, draw BF parallel to AC, meeting either of the adjoining sides AE or CD in F; which point, when the angle ABC is re-entrant will lie within the figure: Join CF. Again, join the alternate points C, E, and through the intermediate point D draw the parallel DG to meet in G either of the adjoining sides AE or BC, and which, since the angle CDE is salient, must for that effect be produced; and join CG. The triangle FCG is equivalent to the five-sided figure ABCDE.



Because the triangles CFA and CBA have by construction the same altitude and stand on the same base AC, they are (II. 1.) equivalent; take each away from the space ACDE, and there remains the quadrilateral figure FCDE equivalent to the five-sided figure ABCDE. Again, because the triangles CDE and CGE are equal, having the same altitude and the same base; add the triangle FCE to each, and the triangle FCG is equivalent to the quadrilateral figure FCDE, and is consequently equivalent to the original figure ABCDE.

In this manner, any polygon may, by successive steps, be reduced to a triangle; for an exterior triangle is always exchanged for another equivalent one, which, attaching itself to either of the adjoining sides, coalesces with the rest of the figure *.

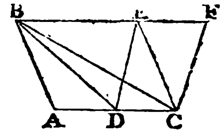
PROP. VI. PROB.

A triangle is equivalent to a rhomboid which has the same altitude and stands on half the base.

The triangle ABC is equivalent to the rhomboid DEFC, which stands on half the base DC, but has the same altitude.

* *See Note XIV.*

For join BD and EC. The triangles ABD and DBC having the same vertex and equal bases, are (II. 2.) equivalent. But the diagonal EC bisects the rhomboid DEFC (I. 27. cor.), and the triangles DBC and DEC, having the same altitude, are equivalent (II. 1.); consequently their doubles, or the triangle ABC and the rhomboid DEFC, are equivalent.

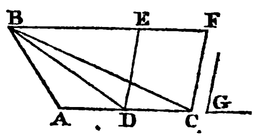


PROP. VII. PROB.

To construct a rhomboid equivalent to a given rectilineal figure, and having its angle equal to a given angle.

Let it be required to construct a rhomboid which shall be equivalent to a given rectilineal figure and contain an angle equal to G.

Reduce the rectilineal figure to an equivalent triangle ABC (II. 5.), bisect the base AC in the point D (I. 7.), and draw DE making an angle CDE equal to the given angle G (I. 4.), through B draw BF parallel to AC (I. 24.), and through C the straight line CF parallel to DE: DEFC is the rhomboid that was required.



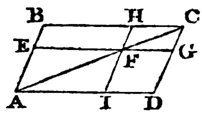
For the figure DF is, by construction, a rhomboid, contains an angle CDE equal to G, and is equivalent to the triangle ABC (II. 6.), and consequently to the given rectilineal figure.

PROP. VIII. THEOR.

The complements of the rhomboids about the diagonal of a rhomboid, are equivalent.

Let EI and HG be rhomboids about the diagonal of the rhomboid BD; their complements BF and FD contain equal spaces.

Since the diagonal AF bisects the rhomboid EI (I. 27. cor.), the triangle AEF is equivalent to AIF; and, for the same reason, the triangle FHC is equivalent to FGC. From the whole triangle ABC on the one side of the diagonal, take away the two triangles AEF and FHC; and from the triangle ADC, which is equal to it, take away, on the other side, the two triangles AIF and FGC, and there remains the rhomboid BF equivalent to FD.

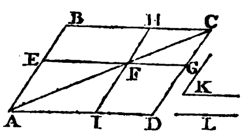


PROP. IX. PROB.

With a given straight line to construct a rhomboid equivalent to a given rectilineal figure, and having an angle equal to a given angle.

Let it be required to construct, with the straight line L, a rhomboid, containing a given space, and having an angle equal to K.

Construct (II. 7.) the rhomboid BF equivalent to the given rectilineal figure, and having an angle BEF equal to K; produce EF until FG be equal to L, through G draw DGC parallel to EB and meeting the extension of BH in C, join CF and produce it to meet the extension of BE in A; draw AD parallel to EF, meeting CG in D, and produce HF to I: FD is the rhomboid required.



For FD and FB are evidently complementary rhomboids, and therefore (II. 8.) equivalent; and, by reason of the parallels AE, IF, the angle FID is equal to EAI (I. 23.), which again is equal to BEF or the given angle K.

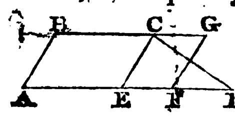
PROP. X. THEOR.

A trapezoid is equivalent to the rectangle contained by its altitude and half the sum of its parallel sides.

The trapezoid ABCD is equivalent to the rectangle contained by its altitude and half the sum of the parallel sides BC and AD.

For draw CE parallel to AB (I. 24.), bisect ED (I. 7.) in F, and draw FG parallel to AB, meeting the production of BC in G.

Because BC is equal to AE (I. 27.), BC and AD are together equal to AE and AD, or to twice AE with ED, or to twice AE and twice EF, that is, to twice AF; consequently AF is half the sum of BC and AD. Wherefore the rectangle contained by the altitude of the trapezoid and half the sum of its parallel sides, is equivalent to the rhomboid BF (II. 1. cor.); but the rhomboid EG is equivalent to the triangle ECD (II. 6.), add to each the rhomboid BE, and the rhomboid BF is equivalent to the trapezoid ABCD *.



PROP. XI. THEOR.

The square described on the hypotenuse of a right-angled triangle, is equivalent to the squares of the two sides.

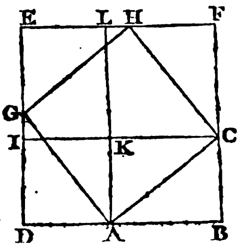
Let ACB be a triangle which is right-angled at B; the square of the hypotenuse AC is equivalent to the two squares of AB and BC.

For produce the base BA until AD be equal to the perpendicular BC, and on DB describe (I. 37.) the square DEFB, make EG and FH equal to AD or BC, join AG, GH, and HC, and through the points A and C (I. 24.) draw AL and CI parallel to BF and BD.

Because the whole line BD is equal to DE, and the part of it AD equal to GE, the remainder AB is equal to DG;

* See Note XV.

wherefore (I. 3.) the triangles ACB and AGD are equal, since they have the sides AB, BC equal to DG, DA, and the contained angle ABC equal to ADG, both of these being right angles. In the same manner, it is proved, that the triangle ACB is equal to GEH, and to HFC. Consequently the sides AC, AG, GH, and HC are all equal. But the angle CAB, being equal to AGD, is equal to the alternate angle GAL (I. 23.); add LAC to each, and the whole angle LAB or (I. 27.) EDB is equal to GAC, which is therefore a right angle. Hence the figure AGHC, having all its sides equal and one of its angles right, is a square.



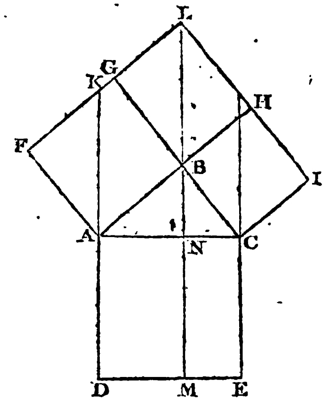
Again, the rhomboids KB and KE are evidently rectangular; they are also equal, being contained by equal sides; and each of them being double of the original triangle ACB, they are together equal to the four triangles ACB, AGD, EHG, and HCF. The other inscribed figures LC and IA are obviously the squares of KC and AD, which are equal to the base and perpendicular of the triangle ABC. From the whole square DEFB, therefore, take away separately those four encompassing triangles with the two interjacent rectangles KB and KE, and the remainders must be equal; that is, the square AGHC is equal in space to both the squares ADIK and KLFC.

Otherwise thus.

Let the triangle ABC be right-angled at B; the square described on the hypotenuse AC is equivalent to BF and BI the squares of the sides AB and BC.

For produce DA to K, and through B draw MBL parallel to DA (I. 24.) and meeting FG produced in L.

Because the angle CAK, adjacent to CAD, is a right angle, it is equal to BAF: from each take away the angle BAK, and there remains the angle BAC equal to FAK. But the angle ABC is equal to AFK, both being right angles. Wherefore the triangles ABC and AFK, having thus two angles of the one respectively equal to those of the other, and the interjacent side AF equal to AB, are equal (I. 21.), and consequently the side AC is equal to AK. Hence the rectangle or rhomboid AM is equivalent to ABLK (II. 2. cor.), since they stand on equal bases AD and AK, and between the same parallels DK and ML. But ABLK is (II. 1. cor.) equivalent to the rhomboid or square BF, for it stands on the same base AB and between the same parallels FL and AH. Wherefore the rectangle AM is equivalent to the square of AB. And in like manner, by drawing MB to meet the production of HI, it may be proved, that the rectangle CM is equivalent to the square of BC. Consequently the whole square, ADEC, of the hypotenuse, contains the same space as both together of the squares described on the two sides AB and BC*.



PROP. XII. THEOR.

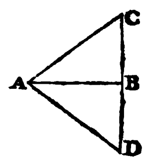
If the square of a side of a triangle be equivalent to the squares of both the other sides, that side subtends a right angle.

Let the square described on AC be equivalent to the two squares of AB and BC; the triangle ABC is right-angled at B.

* See Note XVI.

For draw BD perpendicular to AB (I. 36.) and equal to BC, and join AD.

Because BC is equal to BD, the square of BC is equal to the square of BD, and consequently the squares of AB and BC are equal to the squares of AB and BD. But the squares of AB and BC are, by hypothesis, equivalent to the square of AC; and since ABD is, by construction, a right angle, the squares of AB and BD are (II. 11.) equivalent to the square of AD. Whence the square of AC is equivalent to that of AD, and the straight line AC equal to AD. The two triangles ACB and ADB, having all the sides in the one respectively equal to those in the other, are therefore equal (I. 2.), and consequently the angle ABC is equal to the corresponding angle ABD, that is, to a right angle *.



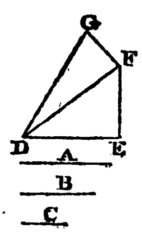
PROP. XIII. PROB.

To find the side of a square equivalent to any number of given squares.

Let A, B, and C be the sides of the squares, to which it is required to find an equivalent square.

Draw DE equal to A, and from its extremity E erect (I. 36.) the perpendicular EF equal to B, join DF, and again perpendicular to this draw FG equal to C, and join DG: DG is the side of the square which was required.

For because DEF is a right-angled triangle, the square of DF is equivalent to the squares of DE and EF (II. 11.), or of A and B. Add the square of FG or C, and the squares of DF and FG, which are equivalent to the square of DG (II. 11.), are equivalent to the aggregate squares of A, B, and C. And by thus repeating the process, it may be extended to any number of squares.



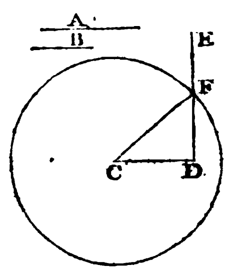
* See Note XVII.

PROP. XIV. PROB.

To find the side of a square equivalent to the difference between two given squares.

Let A and B be the sides of two squares; it is required to find a square equivalent to their difference.

Draw CD equal to the smaller line B, from its extremity erect (I. 36.) the indefinite perpendicular DE, and about the centre C with a distance equal to the greater line A describe a circle cutting DE in F: DF is the side of the square required.



For join CF. The triangle CDF being right-angled, the square of the hypotenuse CF is equivalent to the squares of CD and DF (II. 11.), and consequently taking the square of CD from both, the excess of the square of CF above that of CD is equivalent to the square of DF, or the square of DF is equivalent to the excess of the square of A above that of B.

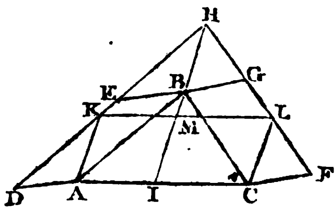
PROP. XV. THEOR.

In any triangle, the rhomboids described on two sides, are together equivalent to a rhomboid described on the base, and limited by these and by parallels to the line which joins the vertex with their point of concourse.

Let ADEB and BGFC be rhomboids described on the two sides AB and BC of the triangle ABC; produce the summits DE and FG to meet in H, join this point with the vertex B, to BH draw the parallels AK, CL, and join KL. It is obvious that AK and CL, being equal and parallel to BH, are

likewise equal and parallel to each other, and that the figure AKLC is a parallelogram or rhomboid.—This rhomboid is equivalent to the two rhomboids BD and BF.

For produce HB to meet the base AC in I. And because the rhomboids KI and AH stand on the same base AK and between the same parallels, they are equivalent (II. 1. cor.); but the rhomboids AH and BD, standing on the same base AB and between the same parallels, are also equivalent. Whence KI is equivalent to BD. And in the same manner, it may be proved that LI is equivalent to BF. Consequently the whole rhomboid KC is equivalent to the two rhomboids BD and BF *.



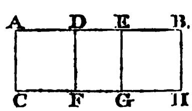
PROP. XVI. THEOR.

The rectangle contained by two straight lines, is equivalent to the rectangles contained under one of them and the several segments into which the other is divided.

The rectangle under AC and AB, is equivalent to the rectangles contained by AC and the segments AD, DE, and EB.

For, through the points D and E, draw DF and EG parallel and equal to AC (I. 24.).

The figures AF, DG, and EH are evidently rhomboidal; they are also rectangular, for the angles ADF, AEG, and ABH are each equal to the opposite angle ACF (I. 21.). And the opposite sides DF, EG, and BH, being equal to AC,—the spaces into which the rectangle BC is resolved, are equal to the rectangles contained by AC and AD, DE and EB †.



* See Note XVIII.

† See Note XIX.

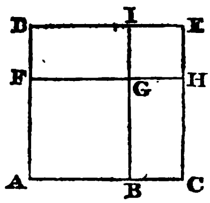
PROP. XVII. THEOR.

The square described on the sum of two straight lines, is equivalent to the squares of those lines, together with twice their rectangle.

If AB and BC be two straight lines placed continuous; the square described on their sum AC, is equivalent to the two squares of AB, BC, and twice the rectangle contained by them.

For through B draw BI (I. 24.) parallel to AD, make AF equal to AB, and through F draw FH parallel to DE.

It is manifest that the spaces AG, GE, DG and CG, into which the square of AC is divided, are all rhomboidal and rectangular. And because AB is equal to AF, and the opposite sides equal, the figure AG is equilateral, and having a right angle at A, is hence a square. Again, AD being equal to AC, take away the equals AF and AB, and there remains DF equal to BC, and consequently IG equal to GH (I. 27.); wherefore IH is likewise a square. The rectangle DG is contained by the sides FG and DF, which are equal to AB and BC; and the rectangle CG is contained by the sides GB and GH, which are likewise equal to AB and BC. Consequently the whole square of AC is composed of the two squares of AB and BC, together with twice the rectangle contained by these lines.

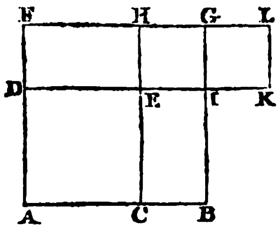


PROP. XVIII. THEOR.

The square described on the difference of two straight lines, is equivalent to the squares of those lines, diminished by twice their rectangle.

Let AC be the difference of two straight lines AB and BC; the square of AC is equivalent to the excess of the two squares of AB and BC above twice their rectangle.

For make AD equal to AC, draw CH and DI (I. 24.) parallel to AF and AB, produce FG until GL be equal to BC, and complete the figure GK.



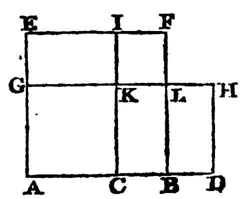
It is evident, from the demonstration of the last Proposition, that DC is the square of AC, and GK the square of BC. From the compound surface AFLKIB, which is made up of the squares of AB and BC, take away twice the rectangle AB, BC, or the two rectangles FI and CG, or the rectangle FI with the rectangle CI and the square IL,—and there remains ADEC, or the square of the difference AC of the two lines AB and BC.

PROP. XIX. THEOR.

The rectangle contained by the sum and difference of two straight lines, is equivalent to the difference of their squares.

Let AB and BD be two continuous straight lines, of which AD is the sum and AC the difference; the rectangle under AD and AC, is equivalent to the excess of the square of AB above that of BC.

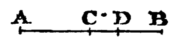
For, having made AG equal to AC, draw GH parallel to AD (I. 24.), and CI, DH parallel to AE.



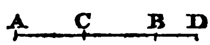
Because GK is equal to KC or HD, and EG is equal to CB or BD, the rectangle EK is equal to LD (II. 2. cor.); and consequently, adding the rectangle BG to each, the space AEIKLB is equivalent to the rectangle AH. But this space AEIKLB is the excess of the square of AB above IL or the square of BC; and the rectangle AH is contained by AD and DH or AC.

Wherefore the rectangle under AD and AC is equivalent to the difference of the squares of AB and BC.

Cor. 1. Hence if a straight line AB be bisected in C and cut unequally in D, the rectangle under the unequal segments AD, DB, together with the square of CD, the interval between the points of section, is equivalent to the square of AC, the half line. For AD is the sum of AC, CD, and DB is evidently their difference; whence, by the Proposition, the rectangle AD, DB is equivalent to the excess of the square of AC above that of CD, and consequently the rectangle AD, DB, with the square of CD, is equal to the square of AC.



Cor. 2. If a straight line AB be bisected in C and produced to D, the rectangle contained by AD the whole line thus produced, and the produced part DB, together with the square of the half line AC, is equivalent to the square of CD, which is made up of the half line and the produced part. For AD is the sum of AC, CD, and DB is their difference; whence the rectangle AD, DB is equivalent to the excess of the square of CD above AC, or the rectangle AD, DB, with the square of AC, is equivalent to the square of CD.



Scholium. If we consider the distances DA, DB of the point D from the extremities of AB as segments of this line, whether formed by *internal* or *external* section; both corollaries may be comprehended under the same enunciation, namely, that if a straight line be divided equally and unequally, the rectangle contained by the unequal segments is equivalent to the difference of the squares of the half line and of the interval between the points of section.

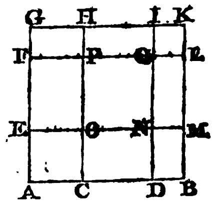
PROP. XX. THEOR.

The square described on a straight line, is equivalent to the squares of the segments into which it is

divided, and twice the rectangles contained by each pair of these segments.

The square of AB is equivalent to the squares of AC, of CD and of DB, with twice the rectangles of AC, CD, of AC, DB, and of CD, DB.

For make AE and EF equal to AC and CD, draw EM, FL parallel to AB, and CH, DI parallel to AG.



It is manifest that AO is the square of AC, OQ the square of CD, and QK the square of DB. Nor is it less obvious that the two rectangles CN and EP are contained by AC, CD, that the two rectangles NL and PI are contained by CD, DB, and that the two rectangles DM and FH are contained by AC, DB. But those squares and those double rectangles complete the whole square of AB. Wherefore the truth of the Proposition is established.

Cor. Hence if a straight line be divided into three portions, the squares of the double segments AD, BC, together with twice the rectangle under the extreme segments AC, BD, are equivalent to the squares of the whole line AB and of the intermediate segment CD. For the squares FD, HM, together with the equal rectangles GP, NB, evidently fill up the whole square AB, with the repetition of the internal square OQ; that is, the squares of AD and BC, with twice the rectangle AC, DB, are equivalent to the squares of AB and CD.

PROP. XXI. THEOR.

The sum of the squares of two straight lines, is equivalent to twice the squares of half their sum and of half their difference.

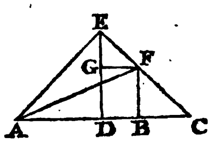
Let AB, BC be two continuous straight lines, D the middle point of AC, and consequently AD half the sum of these lines and DB half their difference; the squares of AB and BC are together equivalent to twice the square of AD with twice the square of DB.

For (II. 17.) the square of AB, or the square of the sum of AD and DB, is equivalent to the squares of these segments, with double their rectangle; and (II. 18.) the square of BC, or that of the difference of AD and DB, is equal to the squares of AD and DB, diminished by double the rectangle contained by the same lines AD, DB. Wherefore the squares of AB and BC taken together, are equivalent simply to twice the squares of AD and DB.

Otherwise thus.

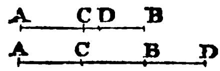
Bisect AC in D (I. 7.), and erect (I. 5. cor.) the perpendicular DE equal to AD or DC, join AE and EC, through B and F draw (I. 24.) BF and FG parallel to DE and AC, and join AF.

Because AD is equal to DE, the angle DAE (I. 11.) is equal to DEA, and since (I. 32. cor.) they make up together one right angle, each of them must be half a right angle. In the same manner, the angles DEC and DCE of the triangle EDC are proved to be each half a right angle; consequently the angle AEC, composed of AED and CED, is equal to a whole right angle. And in the triangle FBC, the angle CBF being equal to CDE (I. 23.) which is a right angle, and the angle BCF being half a right angle—the remaining angle BFC is also half a right angle (I. 32.), and therefore equal to the angle BCF; whence (I. 1.) the side BF is equal to BC. By the same reasoning, it may be shown, that the right-angled triangle GEF is likewise isosceles. The square of the hypotenuse EF, which is equivalent to the squares of EG and



GF (II. 11.) is therefore equivalent to twice the square of GF or of DB; and the square of AE, in the right-angled triangle ADE, is equivalent to the squares of AD and DE, or twice the square of AD. But since ABF is a right angle, the square of AF is equivalent to the squares of AB and BF, or AB and BC; and because AEF is also a right angle, the square of the same line AF is equivalent to the squares of AE and EF, that is, to twice the squares of AD and DB. Wherefore the squares of AB, BC are together equivalent to twice the squares of AD and DB.

Cor. Hence if a straight line AB be bisected in C and cut unequally in D, whether by *internal* or *external* section, the squares of the unequal segments AD and DB are together equivalent to twice the square of the half line AC, and twice the square of CD the interval between the points of division.

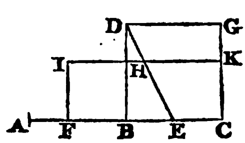


PROP. XXII. PROB.

To cut a given straight line, such that the square of one part shall be equivalent to the rectangle contained by the whole line and the remaining part.

Let AB be the straight line which it is required to divide into two segments, such that the square of the one shall be equivalent to the rectangle contained by the whole line and the other.

Produce AB till BC be equal to it, erect (I. 5. cor.) the perpendicular BD equal to AB or BC, bisect BC in E (I. 7.), join ED and make EF equal to it; the square of the segment BF is equivalent to the rectangle contained by the whole line BA and its remaining segment AF.



For on BC construct the square BG (I. 37.), make BH equal to BF, and draw IHK and FI parallel to AC and BD (I. 24.). Since AB is equal to BD, and BF to BH; the remainder AF is equal to HD: and it is farther evident, that FH is a square, and that IC and DK are rectangles. But BC being bisected in E and produced to F, the rectangle under CF, FB, or the rectangle IC, together with the square of BE, is equivalent to the square of EF or of DE (II. 19. cor. 2.). But the square of DE is equivalent to the squares of DB and BE (II. 11.); whence the rectangle IC, with the square of BE, is equivalent to the squares of DB and BE; or, omitting the common square of BE, the rectangle IC is equivalent to the square of DB. Take away from both the rectangle BK, and there remains the square BI, or the square of BF, equivalent to the rectangle HG, or the rectangle contained by BA and AF.

Cor. 1. Since the rectangle under CF and FB is equivalent to the square of BC, it is evident that the line CF is likewise divided at B in a manner similar to the original line AB. But this line CF is made up, by joining the whole line AB, now become only the larger portion, to its greater segment BF, which next forms the smaller portion in the new compound. Hence this division of a line being once obtained, a series of other lines possessing the same property may readily be found, by repeated additions. Thus, let AB be so cut, that the square of BC is equivalent to the rectangle BA, AC: Make successively BD equal to BA, DE equal to DC, EF

equal to EB, and FG equal to FD; the lines CD, BE, DF, and EG are divided at the points B, D, E, and F, such that, in each of them, the square of the larger part is equivalent to the rectangle contained by the whole and the smaller part.—It is obvious, that this procedure might likewise be reversed.

If FD, EB, and DC be made successively equal to FG, EF and DE, the lines DF, BE, and CD will be divided in the same manner at the points E, D and B.

Cor. 2. Hence also the construction of another problem of the same nature; in which it is required to produce a straight line AB, such that the rectangle contained by the whole line thus produced and the part produced, shall be equivalent to the square of the line AB itself. Divide AB in C, so that the rectangle BA, AC is equivalent to the square of BC, and produce AB until BD be equal to BC: Then, from what has been demonstrated, it follows that the rectangle under AD and DB is equivalent to the square of AB *.

It will be convenient, for the sake of conciseness, to designate in future this remarkable division of a line, where the rectangle under the whole and one part is equivalent to the square of the other, by the term Medial Section.

PROP. XXIII. THEOR.

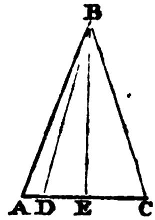
The square of the side of an isosceles triangle is equivalent to the square of a straight line drawn from the vertex to the base, together with the rectangle contained by the segments thus formed.

If BD be drawn from the vertex of the isosceles triangle ABC to a point D in the base; the square of AB is equivalent to the square of BD, together with the rectangle under the segments AD, DC.

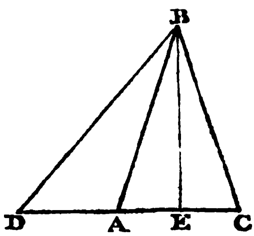
For (I. 7.) bisect the base AC in E, and join BE. Because the triangles ABE and CBE have the sides AB, AE equal to BC, CE, and the side BE common, they are equal (I. 2.),

* *See Note XX.*

and consequently the corresponding angles BEA, BEC are equal, and each of them (Def. 4.) a right angle. Wherefore the square of AB is equivalent to the squares of AE and BE (II. 11.); and since AC is cut equally in E and unequally in D, the square of AE is equivalent to the square of DE, together with the rectangles AD, DC (II. 19. cor. 1.); and consequently the square of AB is equivalent to the squares of BE and DE, together with the rectangle AD, DC. But the square of BD is equivalent to the squares of BE and DE (II. 11.); whence the square of AB is equivalent to the square of BD, together with the rectangle AD, DC.



Cor. The square of a straight line BD drawn from the vertex of an isosceles triangle to any point in the base produced, is equivalent to the square of BA the side of the triangle, together with the rectangle contained by AD and DC, the external segments of the base.



For draw BE, as before, to bisect the base AC. The square of DE is equivalent to the square of AE, together with the rectangle AD, DC (II. 19. cor. 2.); to each of these, add the square of BE, and the squares of DE and BE,—that is, the square of BD (II. 11.)—are equal to the squares of AE and BE, or the square of BA, together with the rectangle AD, DC *.

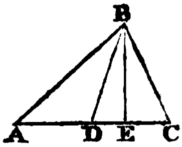
PROP. XXIV. THEOR.

In a scalene triangle, the difference between the squares of the sides, is equivalent to twice the rectangle contained by the base and the distance of its middle point from the perpendicular.

* *See Note XXI.*

Let the side AB of the triangle ABC be greater than BC; and, having let fall the perpendicular BE, and bisected AC in D, the excess of the square of AB above that of BC is equivalent to twice the rectangle contained by AC and DE.

For the square of AB is equivalent to the squares of AE and BE (II. 11.), and the square of BC is equivalent to the squares CE and BE; wherefore the excess of the square of AB above that of BC is equivalent to the excess of the square of AE above that of CE. But the excess of the square of AE above that of CE, is (II. 19.) equivalent to the rectangle contained by their sum AC and their difference, which is evidently the double of DE; and consequently the difference between the squares of AE and CE, being equivalent to the rectangle contained by AC and the double of DE, is equivalent to twice the rectangle under AC and DE.



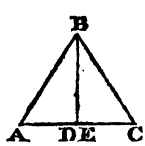
Cor. The difference between the squares of the sides of a triangle, is equivalent to the difference between the squares of the segments of the base made by a perpendicular *.

PROP. XXV. THEOR.

In any triangle, the sum of the squares of the sides, is equivalent to twice the square of half the base and twice the square of the straight line which joins the point of bisection with the vertex.

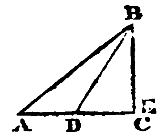
Let BD be drawn from the vertex B of the triangle ABC to bisect the base; the squares of the sides AB and BC are together equivalent to twice the squares of AD and DB.

For let fall the perpendicular BE (I. 6.); and if the point D coincide with E, the triangle ABC being evidently isosceles, the squares of AB and BC are the same with twice the square of AB, or twice the squares of AE and EB, or of AD and DB (II. 11.)

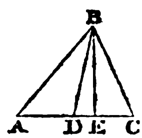


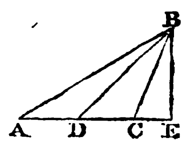
* See Note XXII.

But if the perpendicular fall upon C, the triangle is right-angled, and the squares of AB and BC are then equivalent to the square of AC, and twice the square of BC, or to twice the squares of AD, DC and BC; but (II. 11.) twice the squares of DC and BC are equivalent to twice the square of DB, and consequently the squares of AB and BC are equivalent to twice the squares of AD and DB.



In every other case, whether the perpendicular BE fall within or without the base AC, the squares of AE, EC, the unequal segments of AC, are (II. 21. cor.) equivalent to twice the square of AD and twice the square of DE; add twice the square of EB to both, and the squares of AE, EB and of CE, EB—or the squares of the hypotenuses AB, BC—are equivalent to twice the square of AD, and twice the squares of DE, EB, that is (II. 11.) to twice the square of DB.



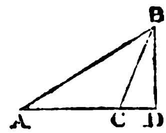


PROP. XXVI. THEOR.

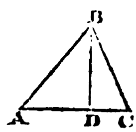
The square of the side of a triangle is greater or less than the squares of the base and the other side, according as the opposite angle is obtuse or acute,—by twice the rectangle contained by the base and the distance intercepted between the vertex of that angle and the perpendicular.

In the oblique-angled triangle ABC, where the perpendicular BD falls without the base; the square of the side AB which subtends the oblique angle exceeds the squares of the sides AC and BC which contain it, by twice the rectangle under AC and CD.

For the square of AD, or of the sum of AC and CD, is (II. 17.) equivalent to the squares of these lines AC, CD, together with twice their rectangle. Add the square of DB to each side, and the squares of AD, DB, or (II. 11.) the square of AB is equivalent to the square of AC, and the squares of CD, DB, together with twice the rectangle AC, CD; but the squares of CD, DB are (II. 11.) equivalent to the square of CB; whence the square of AB exceeds the squares of AC, BC, by twice the rectangle under AC and CD.



Again, in the acute-angled triangle ABC, where the perpendicular BD falls within the triangle; the square of the side AB that subtends the acute angle, is less than the squares of the containing sides AC, BC, by twice the rectangle under the base AC and its intercepted portion CD.



For the square of AD, or of the difference between AC and CD, is (II. 18.) equivalent to the squares of AC and CD, diminished by twice their rectangle. Add to each the square of DB, and the squares of AD and DB—or the square of AB—are equivalent to the square of AC, with the squares of CD and DB, or the square of BC diminished by twice the rectangle under AC and CD. Consequently the square of AB is less than the squares of AC and BC, by twice the rectangle under AC and CD.

Cor. If the side BC be equal to the base AC, the square of the other side AB is equivalent to twice the rectangle under AC and AD, whether the perpendicular BD fall without or within the triangle *.

---

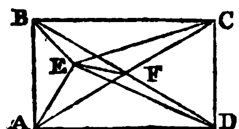
* See Note XXIII.

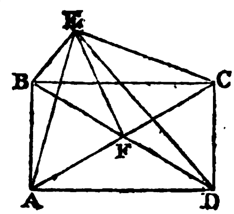
PROP. XXVII. THEOR.

The squares of lines drawn from any point to the opposite corners of a rectangle are together equivalent.

If from a point E, either within or without the rectangle ABCD, straight lines be drawn to the four corners, the squares of AE, EC are together equivalent to the squares of BE, ED.

For join E with F, the intersection of the diagonals AC, BD. Because (I. 29. and cor.) these diagonals are equal, and bisect each other, the lines AF, BF, CF, and DF are all equal. Wherefore the squares of AE, EC are equivalent to twice the square of EF (II. 25.), and the squares of BE, ED are likewise equivalent to twice the square of BF and twice the same square of EF; consequently, the squares of AF and BF being equal, the squares of AE, EC, are together equivalent to the squares of BE, ED *.



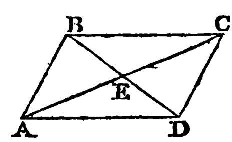


PROP. XXVIII. THEOR.

The squares of the sides of a rhomboid, are together equivalent to the squares of its diagonals.

Let ABCD be a rhomboid: The squares of all the sides AB, BC, CD, and AD, are together equivalent to the squares of the diagonals AC, BD.

For the diagonals bisect each other (I. 31.), and consequently the squares of AB, BC, are equivalent to twice the square of AE and twice the square of BE (II. 30.); wherefore twice the squares of AB, BC, or the squares of all the sides of the



* *See Note XXIV.*

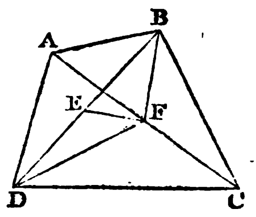
rhomboid, are equivalent to four times the square of AE and four times the square of BE, that is, to the squares of AC and BD.

PROP. XXIX. THEOR.

The squares of the sides of a quadrilateral figure are together equivalent to the squares of its diagonals, together with four times the square of the straight line joining their middle points.

Let ABCD be a quadrilateral figure, in which the straight lines AC, BD, drawn to the opposite corners, are bisected at the points E, F; the squares of AB, BC, CD, and DA, are together equivalent to the squares of AC, BD, together with four times the square of EF.

For join EF. And because AC is bisected in F, the squares of AB and BC are equivalent to twice the square of AF and twice the square of BF (II. 25.); and, for the same reason, the squares of CD and DA are equivalent to twice the square of AF and twice the square of DF. Consequently the squares of all the sides AB, BC, CD, and DA, are equivalent to four times the square of AF—or the square of AC—with twice the square of BF and of DF. But twice these squares of BF and DF is equivalent (II. 21.) to four times the square of BE, or the square of BD, with four times the square of EF; whence the squares of all the sides of the quadrilateral figure are together equivalent to the squares of its diagonals AC, BD, with four times the square of the straight line EF which joins their points of equal section *.



* See Note XXV.

ELEMENTS

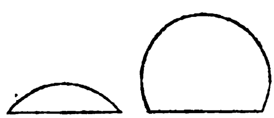
OF

GEOMETRY.

BOOK III.

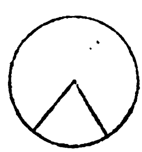
DEFINITIONS.

1. Any portion of the circumference of a circle is called an *arc*, and the straight line which joins the two extremities, a *chord*.

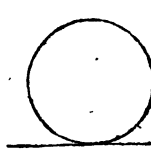


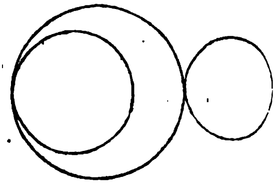
2. The space included between an arc and its chord, is named a *segment*.

3. A *sector* is the portion of a circle contained by two radii and the arc between them.



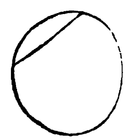
4. The *tangent* to a circle is a straight line which *touches* the circumference, or meets it only in a single point.





5. Circles are said to *touch* mutually, if they meet, but do not cut each other.

6. The point where a straight line touches a circle, or one circle touches another, is called the point of *contact*.



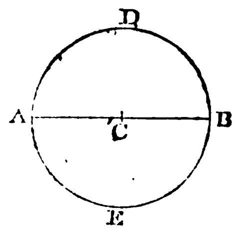
7. A straight line is said to be *inflected* from a point, when it terminates in another straight line, or at the circumference of a circle.

PROP. I. THEOR.

A circle is bisected by its diameter.

The circle ADBE is divided into two equal portions by the diameter AB.

For let the portion ADB be reversed and applied to AEB, the straight line AB and its middle point, or the centre C, remaining the same. And since the radii of the circle are all equal, or the distance of C from any point in the boundary ADB is equal to its distance from any point of the opposite boundary AEB, every point D of the former must meet with a corresponding point of the latter, and consequently the two portions ADB and AEB will entirely coincide.

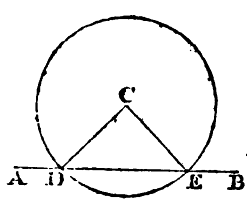


Cor. The portion ADB limited by a diameter, is thus a *semicircle*, and the arc ADB is a *semicircumference*.

PROP. II. THEOR.

A straight line cuts the circumference of a circle only in two points.

If the straight line AB cut the circumference of a circle in D, it can only meet it again in another point E.

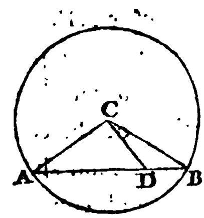


For join D and the centre C; and because from the point C only two equal straight lines, such as CD and CE, can be drawn to AB (I. 18. cor.), the circle described from C through the point D will cross AB again only at E.

PROP. III. THEOR.

The chord of an arc lies wholly within the circle.

The straight line AB which joins two points A, B in the circumference of a circle, lies wholly within the figure.



For, from the centre C, draw CD to any point in AB, and join CA and CB.

Because CDA is the exterior angle of the triangle CDB, it is greater (I. 8.) than the interior CBD or CBA; but CBA, being (I. 11.) equal to CAB or CAD, the angle CDA is consequently greater than CAD, and its opposite side CA (I. 14.) greater than CD, or CD is less than CA, and therefore the point D must lie within the circle.

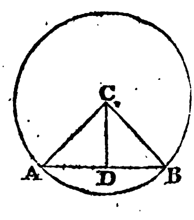
Cor. Hence a circle is concave towards its centre.

PROP. IV. THEOR.

A straight line drawn from the centre of a circle at right angles to a chord, likewise bisects it; and, conversely, the straight line which joins the centre with the middle of a chord, is perpendicular to it.

The perpendicular let fall from the centre C upon the chord AB, cuts it into two equal parts AD, DB.

For join CA, CB: And, in the triangles ACD, BCD, the side AC is equal to CB, CD is common to both, and the right angle ADC is equal to BDC; these triangles, being of the same affection, are equal (I. 22.) and consequently the corresponding side AD is equal to BD.



Again, let AD be equal to BD; the bisecting line CD is at right angles to AB.

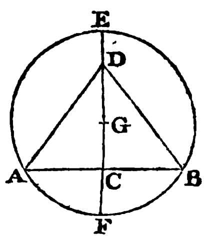
For join CA, CB. The triangles ACD and BCD, having the sides AC, AD equal to CB, BD, and the remaining side CD common to both, are equal (I. 2.), and consequently the angle CDA is equal to CDB, and each of them a right angle.

Cor. Hence a straight line cutting two concentric circles has equal portions intercepted.

PROP. V. THEOR.

A straight line which bisects a chord at right angles, passes through the centre of the circle.

If the perpendicular FE bisect a chord AB, it will pass through G the centre of the circle.



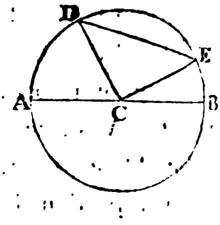
For in FE take any point D, and join DA and DB. The triangles ADC and BDC, having the side AC equal to BC, CD common, and the right angle ACD equal to BCD, are equal (I. 3.), and consequently the base AD is equal to BD. The point D is, therefore, the centre of a circle described through A and B; and thus the centres of the circles that can pass through A and B are all found in the straight line EF. The centre G of the circle AEBF must hence occur in that perpendicular.

Cor. The centre of a circle may hence be found by bisecting the chord AB by the diameter EF (I. 7.), and bisecting this again in G.

PROP. VI. THEOR.

The greatest line that can be inflected within a circle, is the diameter.

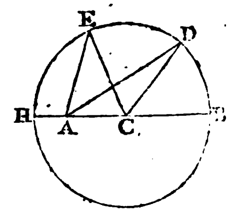
The diameter AB is greater than any chord DE.



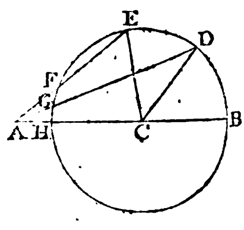
For join CD and CE. The two sides DC and EC of the triangle DCE are together greater than the third side DE (I. 15.); but DC and CE are equal to AC and CB, or to the whole diameter AB. Wherefore AB is greater than DE.

PROP. VII. THEOR.

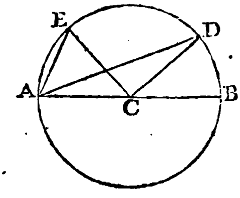
If from any eccentric point, two straight lines be drawn to the circumference of a circle; the one which passes nearer the centre, is greater than that which lies more remote.



Let C be the centre of a circle, and A a different point, from which two straight lines AD and AE are drawn to the circumference; of these lines, AD, which lies nearer to B the opposite extremity of the diameter, is greater than AE.



For the triangles ADC and AEC have the side CD equal to CE, the side CA common to both, but the contained angle DCA greater than ECA; wherefore (I. 19.) the base AD is likewise greater than the base AE.



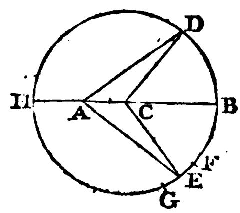
Cor. 1. Hence the straight line ACB, which passes through the centre, is the greatest of all those that can be drawn to the circumference of the circle from the eccentric point A. For it is evident from the Proposition,

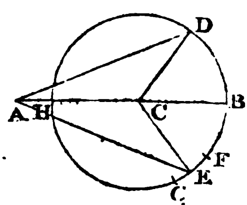
that the nearer the point D approaches to B, the greater is AD; consequently the point B forms the extreme limit of majority, or AB is the greatest line that can be drawn from A to the circumference.

Cor. 2. Hence also, whether the eccentric point be within or without the circle, the straight line AH is the shortest that can be drawn from A to the circumference. For AE is less than AD, and AG less than AF; and the nearer the terminating point approaches to H, which is obviously the most remote from B, the shorter must be its distance from A. Wherefore the point H marks the limit of minority, and AH is the shortest line that can be drawn from A to the circumference of the circle.

PROP. VIII. THEOR.

From any eccentric point, not more than two equal straight lines can be drawn to the circumference, one on each side of the diameter.





Let A be a point which is not the centre of the circle, and AD a straight line drawn from it to the circumference.

Find the centre C (III. 5. cor.) join CA and CD, draw (I. 4.) CE making an angle ACE equal to ACD and cutting the circumference in E, and join AE: The straight lines AE, AD are equal.

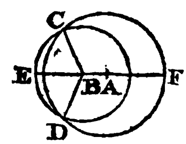
For the triangles ADC, AEC, having the side CD equal to CE, the side AC common, and the contained angle ACD equal to ACE, are equal (I. 3.), and consequently the base AD is equal to AE.

But, except AE, no straight line can be drawn from A on the same side of the diameter HB, that shall be equal to AD: For if the line terminate in a point F between E and B, it will be greater than AE (III. 7.); and if the line terminate in G between E and H, it will, for the same reason, be less than AE.

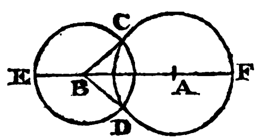
Cor. That point from which more than two equal straight lines can be drawn to the circumference, is the centre of the circle.

PROP. IX. THEOR.

One circle will not cut another in more than two points.



Let DCF and DCE be two circles, of which A and B are the centres; join B with the intersections C and D.



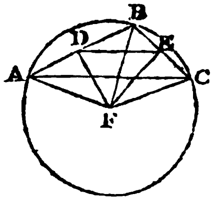
And because B is a point different from the centre A of the circle DCF, not more than two equal straight lines BC and BD can be drawn from it to the circumference of that circle (III. 8.); consequently the circle, described from B as a centre and through the points C and D, will not again meet the circumference DCF.

PROP. X. THEOR.

A circle may be described through three points which are not in the same straight line.

Let A, B, C, be three points not lying in the same direc-

tion; the circumference of a circle may be made to pass through these points.



For (I. 7.) bisect AB by the perpendicular DF, and BC by the perpendicular EF. These straight lines DF, EF will meet; because, DE being joined, the angles EDF, DEF are less than BDF, BEF, and consequently are together less than two right angles, and DF, EF are not parallel (I. 23.) but concur to form a triangle whose vertex is F.

Again, every circle that passes through the two points A and B, has its centre in the perpendicular DF (III. 5.); and, for the same reason, every circle that passes through B and C has its centre in EF; consequently the circle which would pass through all the three points, must have its centre in F, the point common to both perpendiculars DF and EF.

It is manifest, that there is only one circle which can be made to pass through the three points A, B, C; for the intersection of the straight lines DF and EF, which marks the centre, is a single point.

Cor. Hence the mode of describing a circle about a given triangle ABC.

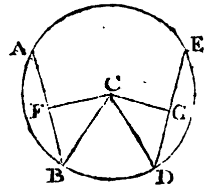
PROP. XI. THEOR.

Equal chords are equidistant from the centre of a circle; and chords which are equidistant from the centre, are likewise equal.

Let AB, DE be equal chords inflected within the same circle; their distances from the centre, or the perpendiculars CF, CG, let fall upon them, are equal.

For the perpendiculars CF and CG bisect the chords AB and DE (III. 4.), and consequently BF, DG, the halves of

these, are likewise equal. The right-angled triangles CBF and CDG, which are thus of the same affection, having the two sides BC, BF equal respectively to DC, DG, and the corresponding angle BFC equal to DGC, are equal (I. 22.), and consequently the side FC is equal to GC.



Again, if the chords AB, DE be equally distant from the centre, they are themselves equal.

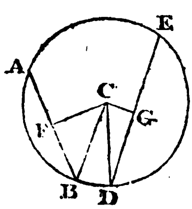
For the same construction remaining: The triangles CBF and CDG are still of the same affection; and have now the two sides CB, CF equal to CD, CG, and the angle BFC equal to DGC; consequently they are equal, and the side BF equal to DG; the doubles of these, therefore, or the whole chords AB, DE, are equal.

PROP. XII. THEOR.

The greater chord is nearer the centre of the circle; and the chord which is nearer the centre is likewise the greater.

Let the chord DE be greater than AB; its distance from the centre, or the perpendicular CG let fall upon it, is less than the distance CF.

For in the right-angled triangle BCF, the square of the hypotenuse BC is equivalent to the squares of BF and FC (II. 11.); and, for the same reason, the square of the hypotenuse DC of the right-angled triangle DCG is equivalent to the squares of DG and GC. But BC and DC are equal, and consequently their squares; wherefore the squares of DG and GC are equivalent to the squares of BF and FC. And since DE is greater than AB, its half DG is greater than BF, and consequently the square



of DG is greater than the square of BF; the square of GC is, therefore, less than the square of FC, because, when joined to the squares of DG and BF, they produce the same amount, or the square of the radius of the circle. Hence the perpendicular GC itself is less than FC.

Again, if the chord DE be nearer the centre than AB, it is also greater.

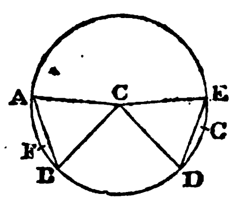
For the same construction remaining: It is proved that the squares of BF and FC are together equal to the squares of DG and GC; but GC being less than FC, the square of GC is less than the square of FC, and consequently the square of DG is greater than the square of BF; whence the side DG is greater than BF, and its double, or the chord DE, greater than AB.

PROP. XIII. THEOR.

In the same or equal circles, equal angles at the centre are subtended by equal chords, and terminated by equal arcs.

If the angle ACB at the centre C be equal to DCE, the chord AB is equal to DE, and the arc AFB is equal to DGE.

For let the sector ACB be applied to DCE. The centre remaining in its place, the radius CA will lie on CD; and the angle ACB being equal to DCE, the radius CB will adapt itself to CE. And because all the radii are equal, their extreme points A and B must coincide with D and E; wherefore the straight lines which join those points, or the chords AB and DE, must coincide. But the arcs AFB and DGE that connect the same points, will also coincide; for any intermediate point F in the one, being at the same distance



from the centre as every point of the other, must, on its application, find always a corresponding point G.

The same mode of reasoning is applicable to the case of equal circles.

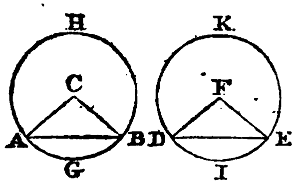
Cor. Hence, in the same or equal circles, equal arcs are subtended by equal chords, and terminate equal angles at the centre.

PROP. XIV. THEOR.

In the same or equal circles, equal chords subtend equal arcs of a like kind.

If the chord AB be different from the diameter, it will evidently subtend at the same time two unequal portions of the circumference of a circle, the one terminating the angle ACB at the centre and less than the semicircumference, the other greater than this and terminating the reversed angle.

In the equal circles GAHB and IDKE the chord AB subtends the arcs AGB and AHB, which are respectively equal to DIE and DKE subtended by the equal chord DE.



For join CA, CB, and FD, FE. The two triangles CAB and FDE, having all the sides of the one equal to those of the other, are equal (I. 2.); and consequently the angle ACB is equal to DFE. Wherefore the arcs AGB and DIE, which terminate these equal angles, are (III. 13.) themselves equal; and hence the remaining portions AHB and DKE of the equal circumferences are likewise equal.

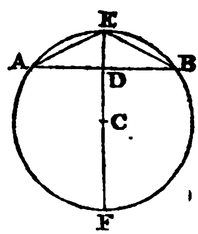
This demonstration, it is evident, will likewise apply in the case of the same circle.

PROP. XV. PROB.

To bisect an arc of a circle.

Let it be required to divide the arc AEB into two equal portions.

Draw the chord AB, and bisect it (I. 7.) by the perpendicular EF cutting the arc AB in E: The arc AE is equal to EB.



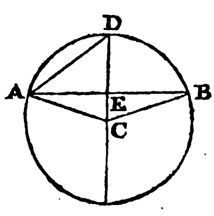
For the triangles ADE, BDE, have the side AD equal to BD, the side DE common, and the containing right angle ADE equal to BDE; they are (I. 3.) consequently equal, and the base AE equal to BE. But these equal chords AE, BE must subtend equal arcs of a like kind (III. 14.), and the arcs AE, BE are evidently each of them less than a semicircumference.

Cor. The correlative arc AFB is also bisected by the perpendicular EF.

PROP. XVI. PROB.

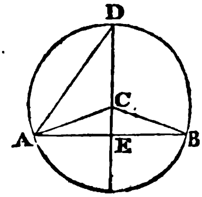
Given an arc, to complete its circle.

Let ADB be an arc; it is required to trace the circle to which it belongs.



Draw the chord AB, and bisect it by the perpendicular CD (I. 7.) cutting the arc in D, join AD, and from A draw AC making an angle DAC equal to ADC (I. 4.): The intersection C of this straight line with the perpendicular, is the centre of the circle required.

For join CB. The triangles ACE and BCE, having the side EA equal to EB, the side EC common, and the contained angle AEC equal to BEC, are equal (I. 3.), and consequently AC is equal to BC. But (I. 12.) AC is also equal to CD, because the angle DAC was made equal to ADC. Wherefore (III. 8. cor.) the three straight lines CA, CD, and CB being all equal, the point C is the centre of the circle.



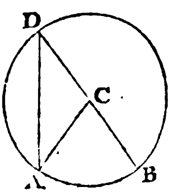
PROP. XVII. THEOR.

The angle at the centre of a circle is double of the angle which, standing on the same arc, has its vertex in the circumference.

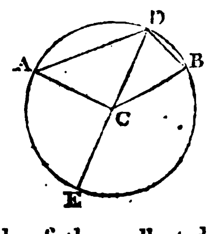
Let AB be an arc of a circle; the angle which it terminates at the centre, is double of ADB the corresponding angle at the circumference.

For join DC and produce it to the opposite circumference. This diameter DCE, if it lie not on one of the sides of the angle ADB, must either fall within that angle or without it.

First, let DC coincide with DB. And because AC is equal to DC, the angle ADC is equal to DAC (I. 11.); but the exterior angle ACB is equal to both of these (I. 32.) and therefore equal to double of either, or the angle ACB at the centre is double of the angle ADB at the circumference.

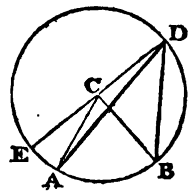


Next, let the straight line DCE lie within the angle ADB. From what has been demonstrated, it is apparent, that the angle ACE is double of ADE, and the angle BCE double of BDE; wherefore the angles ACE, BCE taken together, or the whole angle ACB, are double of the collected



angles ADE, BDE, or the angle ADB at the circumference.

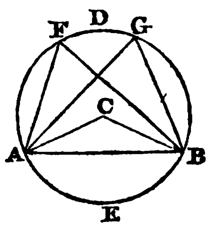
Lastly, let DCE fall without the angle ADB. Because the angle BCE is double of BDE, and the angle ACE is double of ADE; the excess of BCE above ACE, or the angle ACB at the centre, is double of the excess of BDE above ADE, that is, of the angle ADB at the circumference*.



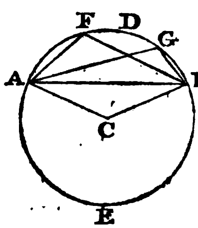
PROP. XVIII. THEOR.

The angles in the same segment of a circle are equal.

Let ADB be the segment of a circle; the angles AFB, AGB contained in it, or which stand on the opposite portion AEB of the circumference, are equal to each other.



For join CA, CB. The angle ACB at the centre is double of the angle AFB or AGB at the circumference (III. 17.); these angles AFB, AGB, which stand on the same arc AEB, are, therefore, the halves of the same central angle ACB, and are consequently equal to each other.



Cor. Hence equal angles at the circumference must stand on equal arcs; for their doubles or the central angles, being equal, are terminated by equal arcs (III. 13.) Hence also equal angles that stand on the same base, have their vertices in the same segment of a circle.

---

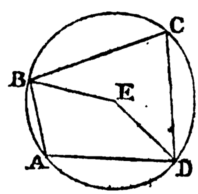
* *See Note XXVI.*

PROP. XIX. THEOR.

The opposite angles of a quadrilateral figure contained within a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure described in a circle; the angles A and C are together equal to two right angles, and so are those at B and D.

For join EB and ED. The angle BED at the centre is double of the angle BCD at the circumference (III. 17.); and for the same reason, the reversed angle BED is double of BAD. Consequently the angles BCD and BAD are the halves of angles about the point E, and which make up four right angles; wherefore the angles BCD and BAD are together equal to two right angles.



In the same manner, by joining EA and EC, it may be proved, that the angles ABC and ADC are together equal to two right angles.

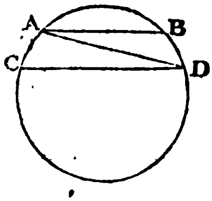
Cor. Hence a circle may be described about a quadrilateral figure which has its opposite angles equal to two right angles; for if a circle be made to circumscribe the triangle BCD (III. 10. cor.), the angles opposite to the base BD are equal to two right angles, and therefore equal to the angles BCD and BAD; consequently the angle BAD is equal to an angle in the segment BAD, and hence (III. 18. cor.) they are contained in the same segment, or the circumference of the circle passes through all the four points A, B, C, and D.

PROP. XX. THEOR.

Parallel chords intercept equal arcs of a circle.

Let the chord AB be parallel to CD; the intercepted arc AC is equal to BD.

For join AD. And because the straight lines AB and CD are parallel, the alternate angles BAD and ADC are equal (I. 23.); wherefore these angles, having their vertices in the circumference of the circle, must stand on equal arcs (III. 18. cor.), and consequently the arcs AC and BD are equal to each other.



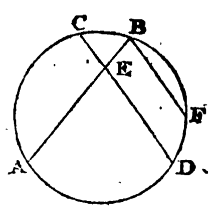
Cor. Hence, conversely, the straight lines which intercept equal arcs of a circle are parallel; and hence another mode of drawing a parallel through a given point to a given straight line *.

PROP. XXI. THEOR.

The inclination of two straight lines is equal to the angle terminated at the circumference by the sum or difference of the arcs which they intercept, according as their vertex is within or without the circle.

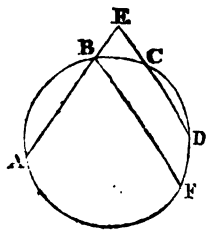
If the two straight lines AB and CD intersect each other in the point E within a circle; the angle AED which they form, is equal to an angle at the circumference and standing on the sum of the intercepted arcs AD and BC.

For draw the chord BF parallel to CD (III. 20. cor.). Because ED and BF are parallel, the angle AED (I. 23.) is equal to the interior angle ABF, which stands on the arc AF; but since the chords BF and CD are parallel, the arc BC is equal to DF (III. 20.) and consequently the arc AF, which terminates at the circumference an angle equal to AED, is the sum of the two intercepted arcs AD and BC.



* *See Note XXVII.*

Again, if the straight lines AB and CD meet at E, without the circle, their inclination AED is equal to an angle at the circumference, having for its base the excess of the arc AD above BC.



For BF being drawn parallel to CD, the arc BC is equal to FD, and consequently the arc AF is the excess of AD above BC; but the angle ABF which stands on AF, is equal to the interior angle AED.

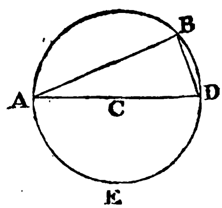
Cor. Hence if two chords intersect each other at right angles within a circle, the opposite intercepted arcs are equal to the semicircumference *.

PROP. XXII. THEOR.

The angle in a semicircle is a right angle, the angle in a greater segment is acute, and the angle in a smaller segment is obtuse.

Let ABD be an angle in a semicircle, or that stands on the semicircumference AED; it is a right angle.

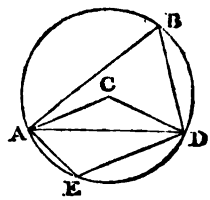
For ABD, being an angle at the circumference, is half of the angle at the centre on the same base AED (III. 17.); it is, therefore, half of the angle ACD formed by the opposite portions CA, CD of the diameter, or half of two right angles, and is consequently equal to one right angle.



Again, let ABD be an angle in a segment greater than a semicircle, or which stands on a less arc AED than the semicircumference; it is an acute angle.

* See Note XXVIII.

For join CA, CD. The angle ABD is half of the central angle ACD, which is evidently less than two right angles; wherefore ABD is less than one right angle, or it is acute.



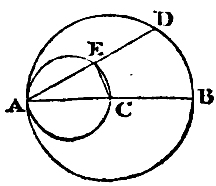
But the angle AED, in the smaller segment, is obtuse. For AED stands on the arc ABD, which is greater than a semicircumference, and is the base of an angle at the centre, the reverse of ACD, and greater, therefore, than two right angles; AED is hence an obtuse angle.

Scholium. From the remarkable property, that the angle in a semicircle is a right angle, may be derived an elegant method of drawing perpendiculars *.

PROP. XXIII. THEOR.

If a circle be described on the radius of another circle, any straight line drawn from the point where they meet to the outer circumference, is bisected by the interior one.

Let AEC be a circle described on the radius AC of the circle ADB, and AD a straight line drawn from A to terminate in the exterior circumference; the part AE in the smaller circle is equal to the part ED intercepted between the two circumferences.



For join CE. And because AEC is a semicircle, the angle contained in it is a right angle (III. 22.); consequently the straight line CE, drawn from the centre C, is perpendicular to the chord AD, and therefore (III. 4.) bisects it.

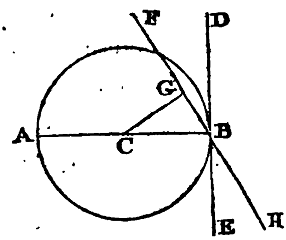
* *See Note XXIX.*

PROP. XXIV. THEOR.

The perpendicular at the extremity of a diameter is a tangent to the circle, and the only tangent which can be applied at that point.

Let ACB be the diameter of a circle, to which the straight line EBD is drawn at right angles from the extremity B; it will touch the circumference at that point.

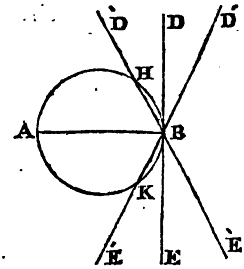
For CB, being perpendicular, is the shortest distance of the centre C from the straight line EBD (I. 18.); wherefore every other point in this line is farther from the centre than B, and consequently falls without the circle.



But the perpendicular EBD is the only straight line which can be drawn through the point B that will not cut the circle. For if HBF were such a line, the perpendicular CG, let fall upon it from the centre, would be less than CB (I. 18.), and would therefore lie within the circle; consequently HBG, being extended, would again meet the circumference, before it effected its escape.

Cor. Hence a straight line drawn from the point of contact at right angles to a tangent, must be a diameter, or pass through the centre of the circle.

Scholium. The nature of a tangent to the circle is easily discovered from the consideration of limits. For suppose the straight line DE, extending both ways, to turn about the extremity B of the diameter AB; it will cut the circle first on the one side of AB, and afterwards on the other. But the arc AH being less than a semicircumference, the angle HBA which the line D'E' makes with the diameter is acute (III. 22.); and, for the same reason, the angle KBA is acute,



and consequently its adjacent angle D'BA is obtuse. Thus the revolving line DE, when it meets the semicircumference AHB, makes an acute angle with the diameter; but when it comes to meet the opposite semicircumference, it makes an obtuse angle. In passing, therefore, through all the intermediate gradations from minority to majority, the line DE must find a certain individual position in which it is at right angles to the diameter, and cuts the circle neither on the one side nor the other.

A similar inference might be derived from Prop. 20. of this Book; one of the parallel chords being supposed to contract, until its extreme points are about to coalesce in the position of the tangent.

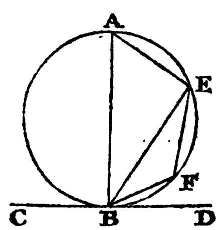
PROP. XXV. THEOR.

If, from the point of contact, a straight line be drawn to cut the circumference, the angles which it makes with the tangent are equal to those in the alternate segments of the circle.

Let CD be a tangent, and BE a straight line drawn from the point of contact, cutting the circle into two segments BAE and BFE; the angle EBD is equal to EAB, and the angle EBC to EFB.

For draw BA perpendicular to CD (I. 5. cor.), join AE, and, from any point F in the opposite arc, draw FB and FE.

Because BA is perpendicular to the tangent at B, it is a diameter (III. 24. cor.), and consequently AEFB is a semicircle; wherefore AEB is a right angle (III. 22.), and the remaining acute angles BAE, ABE of the triangle, being together equal to another right angle, are equal to ABE and EBD, which compose the right angle ABD. Take the angle ABE away from both, and the angle BAE remains equal to EBD



Again, the opposite angles BAE and BFE of the quadrilateral figure BAEF, being equal to two right angles (III. 19.), are equal to the angle EBD with its adjacent angle EBC; and taking away the equals BAE and EBD, there remains the angle BFE equal to EBC.

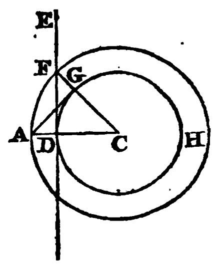
Cor. If a straight line meet the circumference of a circle, and make an angle with an inflected line equal to that in the alternate segment, it touches the circle.

PROP. XXVI. PROB.

To draw a tangent to a circle, from a given point without it.

Let A be a given point, from which it is required to draw a straight line that shall touch the circle DGH.

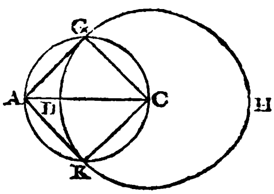
Find the centre C (III. 5. cor.), join CA and draw DE (I. 5. cor.), perpendicular to CA, from C with the distance CA describe a circle meeting DE in F, join CF cutting the interior circumference in G; AG being joined, is the tangent which was required.



For the triangles ACG and FCD have the sides CA, CG equal to CF, CD, and the containing angle ACF common to both; they are, therefore, equal (I. 3.), and consequently the angle CGA is equal to CDF. But CDF is a right angle; whence CGA is likewise a right angle, and AG a tangent to the circle (III. 24.)

Or thus.

On AC as a diameter describe the circle AGCK, cutting the given circle in the points G, K: Join AG, AK; either of these lines is the tangent required.



For join CG, CK. And the angles CGA, CKA, being each in a semicircle, are right angles

(III. 22.), and consequently AG, AK, touch the circle DGHK at the points G, K (III. 24.).

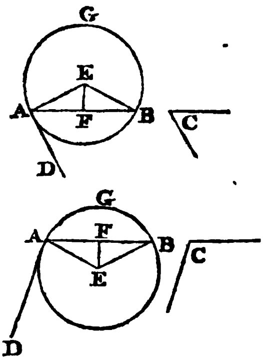
Cor. Hence tangents drawn from the same point to a circle are equal; for the triangles ACG and ACK having the side CG equal to CK, CA common, and the angles at G and K right, are equal (I. 22.), and consequently AG is equal to AK.

PROP. XXVII. PROB.

On a given straight line, to describe a segment of a circle, that shall contain an angle equal to a given angle.

Let AB be a straight line, on which it is required to describe a segment containing an angle equal to C.

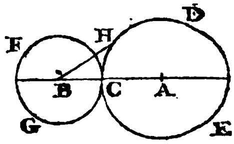
If C be a right angle, it is evident that the problem will be performed, by describing a semicircle on AB. But if the angle C be either acute or obtuse; draw AD (I. 4.) making an angle BAD equal to C (I. 36.), erect AE perpendicular to AD, draw EF (I. 5. cor.) to bisect AB at right angles and meeting AE in E, and, from this point as a centre and with the distance EA describe the required segment AGB.



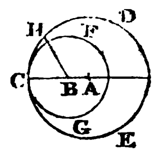
Because EF bisects AB at right angles, the circle described through A must also pass through (III. 5.) the point B; and since EAD is a right angle, AD touches the circle at A (III. 24.), and the angle BAD, which was made equal to C, is equal (III. 25.) to the angle in the alternate segment AGB.

PROP. XXVIII. THEOR.

Two circles which meet in the straight line joining their centres or in its continuation, touch each other.



Let the circles DCE, FCG meet at C, in the direction of the straight line which joins their centres A, B; they touch each other at that point.



For draw BH to another point H in the circumference DCE. And because B is distinct from the centre A, the line BH is greater than BC (III. 7. cor. 2.), and consequently the point H lies without the circle FCG. Except, therefore, at the single point C, the circumference DCE does not meet FCG.

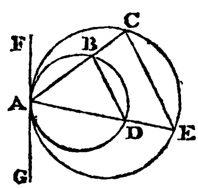
Cor. Hence a straight line extending through the centres of two circles, will pass through their points of contact.

PROP. XXIX. THEOR.

Two straight lines drawn through the point of contact of two circles, intercept arcs of which the chords are parallel.

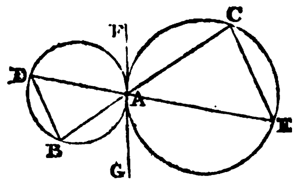
Let the circles ACE and ABD touch mutually in A, and from this point the straight lines AC, AE be drawn to cut the circumferences; the chords CE and BD are parallel.

For draw the tangent FAG, (III. 26.), which must touch both circles.



In the case of internal contact, the angle GAE is equal to ACE in the alternate segment, (III. 25.); and, for the same reason, GAE or GAD is equal to ABD; consequently the angles ACE and ABD are equal, and therefore (I. 23.) the straight lines CE and BD are parallel.

When the contact is external, the angle GAE is still equal to ACE, and its vertical angle FAD is, for the same reason, equal to ABD; whence ACE is equal to ABD; and these being alternate angles, the straight line CE (I. 23.) is parallel to BD.

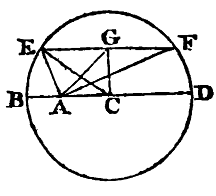


PROP. XXX. THEOR.

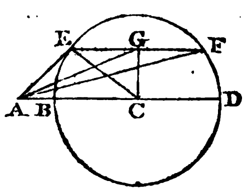
If from any point in the diameter of a circle or its extension, straight lines be drawn to the ends of a parallel chord; the squares of these lines are together equivalent to the squares of the segments into which the diameter is divided.

Let BEFD be a circle, and from the point A in its extended diameter the straight lines AE and AF be drawn to the ends of the parallel chord EF; the squares of AE and AF are together equivalent to the squares of AB and AD.

For from the centre C, let fall the perpendicular CG upon EF (I. 6.), and join AG and CE.



Because CG cuts the chord EF at right angles, GE is equal to GF (III. 4.); wherefore the squares of AE and AF are equivalent to twice the squares of AG and GE (II. 39.) But ACG being a right-angled triangle, the square of AG is equivalent to the squares of AC and CG (II. 11.), or twice the square of AG is equivalent to twice the squares of AC and CG. Wherefore the squares of AE and AF are equivalent to twice the three squares of AC, CG, and GE. Of these, the two squares of CG and GE are equivalent to the square of CE or CB, for



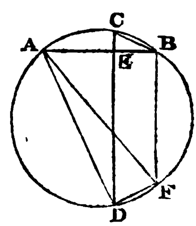
the triangle CGE is right-angled. Consequently the squares of AE and AF are equivalent to twice the squares of AC and CB. But the straight line BD being cut equally at C and unequally at A, the squares of the unequal segments AB and AD are together equivalent to twice the squares of AC and CB (II. 21. cor.); whence the squares of AE and AF are together equivalent to the squares of AB and AD.

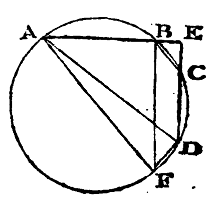
PROP. XXXI. THEOR.

If through a point, within or without a circle, two perpendicular lines be drawn to meet the circumference, the squares of all the intercepted distances are together equivalent to the square of the diameter.

Let E be a point within or without the circle, and AB, CD two straight lines drawn through it at right angles to the circumference; the squares of the four segments EA, EB, ED, and EC, are together equivalent to the square of the diameter of the circle.

For draw BF parallel to CD, and join AF, AD, CB, and DF.





Because BF is parallel to CD, the arc BC is equal to the arc FD (III. 20.), and consequently the chord BC is also equal to the chord FD (III. 15. cor.); but BC being the hypotenuse of the right-angled triangle BEC, its square, or that of FD is equivalent to the squares of EB and EC (II. 11.), and AED being likewise right-angled, the square of AD is equivalent to the squares of EA and ED. Whence the squares of AD and FD are equivalent to the four squares of EA, EB, ED, and EC. But since ED is parallel to BF, the interior angle ABF is equal to AED (I. 23.), and therefore a right angle; consequently ACBF is

a semicircle (III. 23. cor.) and AF the diameter. The angle ADF in the opposite semicircle is hence a right angle (III. 22.), and the square of the diameter AF is equal to the squares of AD and FD, or to the sum of the squares of the four segments EA, EB, ED, and EC intercepted between the circumference and the point E.

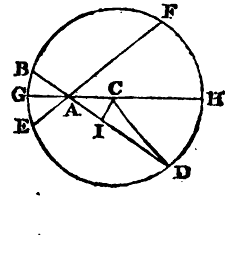
PROP. XXXII. THEOR.

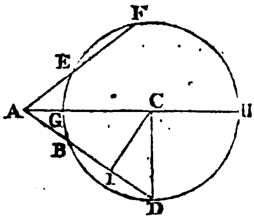
If through a point, within or without a circle, two ~~straight lines~~ be drawn ~~to cut the circumference~~; the rectangle under the segments of the one, is equivalent to that contained by the segments of the other.

Let the two straight lines AD and AF be extended through the point A, to cut the circumference BFD of a circle; the rectangle contained by the segments AE and AF of the one, is equivalent to the rectangle under AB and AD, the distances intercepted from A in the other.

For draw AC to the centre, and produce it both ways to terminate in the circumference at G and H; let fall the perpendicular CI upon BD (I. 6.), and join CD.

Because CI is perpendicular to AD, the difference between the squares of CA and CD, the sides of the triangle ACD is equivalent to the difference between the squares of the segments AI and ID the segments of the base (II. 21. cor.); and the difference between the squares of two straight lines being equivalent to the rectangle under their sum and their difference (II. 19.), the rectangle contained by the

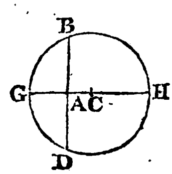




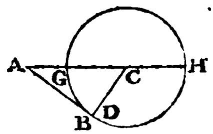
sum and difference of AC, CD is equivalent to the rectangle contained by the sum and difference of AI, ID. But since the radius CG is equal to CH, the sum of AC and CD is AH, and their difference is AG; and because the perpendicular CI bisects the chord BD (III. 4.), the sum of AI and ID is AD, and their difference AB. Wherefore the rectangle AH, AG is equivalent to the rectangle AB, AD. In the same way it is proved, that the rectangle AH, AG is equivalent to the rectangle AE, AF; and consequently the rectangle AE, AF is equivalent to the rectangle AB, AD.

Or thus.

Draw the diameter GAH, and join CB and CD. And because BCD is an isosceles triangle and CA is drawn from the vertex C to a point in the direction of its base, the difference between the square of CA and CD or CG is equivalent to the rectangle contained by the segments AB, AD of the base (II. 24. cor.). In like manner, it is proved that the same difference between the square of CA and CG is equivalent to the rectangle contained by the segments AE, AF; whence the rectangle under AB, AD is equivalent to the rectangle under AE, AF.



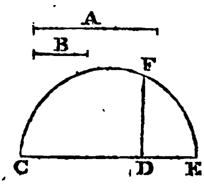
Cor. 1. If the vertex A of the straight lines lie within the circle and the point I coincide with it, BD, being then at right angles to CA, is bisected at A (III. 4.), and the rectangle AB, AD is the same as the square of AB. Consequently the square of a perpendicular AB limited by the circumference, is equivalent to the rectangle under the segments AG, AH of the diameter.



Cor. 2. If the vertex A lie without the circle, and the point I coincide with B or D, the angle ABC being then a right angle, the incident line AB must be a tangent (III. 24.), and consequently the two points of section B and D must coalesce in a single point of contact. Wherefore the rectangle under the distances AB, AD becomes the same as the square of AB; and consequently the rectangle contained by the segments AG, AH of the diameter, is equivalent to the square of the tangent AB.

PROP. XXXIII. PROB.

To construct a square equivalent to a given rectilineal figure.



Let the rectilineal figure be reduced by Proposition 7. Book II. to an equivalent rectangle, of which A and B are the two containing sides; draw an indefinite straight line CE, in which take the part CD equal to A and DE to B, on CE describe a semicircle, and erect the perpendicular DF from the diameter to meet the circumference: DF is the side of the square equivalent to the given rectilineal figure.

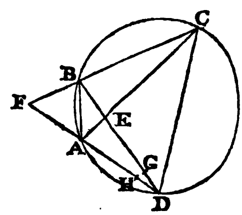
For, by Cor. 1. to the last Proposition, the square of the perpendicular DF is equivalent to the rectangle under the segments CD, DE of the diameter, and is consequently equivalent to the rectangle contained by the sides A and B of a rectangle that was made equivalent to the rectilineal figure.

PROP. XXXIV. THEOR.

A quadrilateral figure may have a circle described about it, if the rectangles under the segments made

by the intersection of its diagonals be equivalent, or if those rectangles are equivalent which are contained by the external segments formed by producing its opposite sides.

Let ABCD be a quadrilateral figure, of which AC and BD are the diagonals, and such that the rectangle AE, EC is equivalent to the rectangle BE, ED; a circle may be made to pass through the four points A, B, C, and D.



For describe a circle through the three points A, B, C (III. 10. cor.), and let it cut BD in G. Because AC and BG intersect each other within a circle, the rectangle AE, EC is equivalent to the rectangle BE, EG (III. 31.); but the rectangle AE, EC is by hypothesis equivalent to the rectangle BE, ED. Wherefore BE, EG is equivalent to BE, ED; and these rectangles have a common base BE, consequently (II. 3. cor.) their altitudes EG and ED are equal, and hence the point G is the same as D, or the circle passes through all the four points A, B, C, and D.

Again, if the opposite sides CB and DA be produced to meet at F, and the rectangle CF, FB be equal to DF, FA, a circle may be described about the figure.

For, as before, let a circle pass through the three points A, B, C, but cut AD in H. And from the property of the circle, the rectangle CF, FB is equivalent to HF, FA; but the rectangle CF, FB is also equivalent to DF, FA; whence the rectangle HF, FA is equivalent to DF, FA, and the base HF equal to DF, or the point H is the same as D *.

* *See Note XXX.*

ELEMENTS

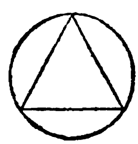
OF

GEOMETRY.

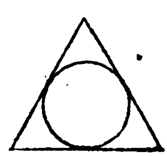
BOOK IV.

DEFINITIONS.

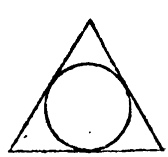
1. A rectilineal figure is said to be *inscribed* in a circle, when all its angular points lie on the circumference.



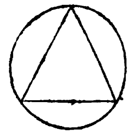
2. A rectilineal figure *circumscribes* a circle, when each of its sides is a tangent.



3. A circle is *inscribed* in a rectilineal figure, when it touches all the sides.



4. A circle is *described* about a rectilineal figure or *circumscribes* it, when the circumference passes through all the angular points of the figure.



5. Polygons are *equilateral*, when their sides, in the same order, are respectively equal: They are *equiangular*, if an equality obtains between their corresponding angles.

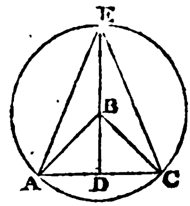
6. Polygons are said to be *regular*, when all their sides and their angles are equal.

PROP. I. PROB.

Given an isosceles triangle, to construct another on the same base, but with half the vertical angle.

Let ABC be an isosceles triangle standing on AC; it is required, on the same base, to construct another isosceles triangle, that shall have its vertical angle half of the angle ABC.

Bisect AC in D (I. 7.), join DB, which produce till BE be equal to BA or BC, and join AE, CE: AEC is the isosceles triangle required.

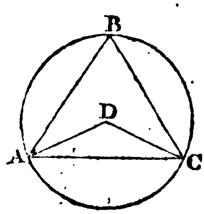


For, the straight line BE being equal to BA and BC, the point B is the centre of a circle which passes through A, E, and C; and consequently the angle ABC is double of AEC at the circumference (III. 17.), or the vertical angle AEC is half of ABC. But the triangles AED and CED, having the side DA equal to DC, the side DE common to both, and the right angle ADE (III. 4.) equal to CDE are (I. 3.) equal, and consequently AE is equal to CE. Wherefore the triangle AEC is likewise isosceles.

PROP. II. PROB.

Given an acute-angled isosceles triangle, to construct another on the same base, which shall have double the vertical angle.

Let ABC be an acute-angled isosceles triangle; it is required, on the base AC, to construct another isosceles triangle, having its vertical angle double of the angle ABC.



Describe a circle through the three points A, B, and C (III. 10. cor.), and draw AD, CD to the centre D; the triangle ADC is the isosceles triangle re-

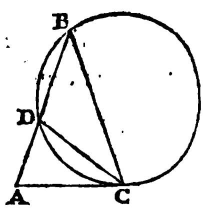
quired. For the angle ADC, being at the centre of the circle, is (III. 17.) double of ABC, the angle at the circumference.

PROP. III. THEOR.

If an isosceles triangle have each angle at the base double of the vertical angle, its base will be equal to the greater segment of one of its sides divided by a medial section.

Let ABC be an isosceles triangle which has each of the angles BAC, BCA double of the vertical angle ABC; the base AC is equal to the greater segment of the side BA formed by a medial section.

For draw CD to bisect the angle BCA (I. 5.), and about the triangle BDC describe a circle (III. 10. cor.).



Because the angle BCA is double of ABC and has been bisected by CD, the angles ACD, BCD are each of them equal to CBD, and consequently the side BD is equal to CD (I. 12.). But the triangles BAC and DAC, having the angle ACD equal to ABC, and the angle at A common to both, must have also (I. 32.) the remaining angle CDA equal to BCA or CAD; whence (I. 12.) the triangle DAC is likewise isosceles, and the side AC equal to CD; and CD being equal to BD, therefore AC is equal to BD. And since the angle ACD is equal to CBD in the alternate segment of the circle, the straight line AC touches the circumference at C (III. 25. cor.); wherefore the rectangle contained by AB and AD (III. 31. cor. 2.) is equivalent to the square of AC, or the square of BD. Consequently the base AC of this isosceles triangle is equal to the greater segment BD of the side AB cut by a medial section.

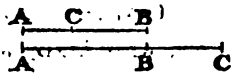
Cor. Hence the interior triangle ACD is likewise isosceles and of the same nature with ABC, having the greater segment of AB for its side, and the smaller segment for its base.

PROP. IV. PROB.

Given either one of the sides or the base, to construct an isosceles triangle, so that each of the angles at the base may be double of its vertical angle.

First, let one of the sides AB be given, to construct such an isosceles triangle.

Divide AB by a medial section at C (II. 22.), and on CB, as a base with the distance AB for each of the sides, describe an isosceles triangle (I. 1.)



Next, let the base AB be given, to construct an isosceles triangle of this nature.

Produce AB to C, such that the rectangle AC, CB be equal to the square of AB (II. 22. cor. 2.), and on the base AB, with the distance AC for each of the sides, describe an isosceles triangle.

These isosceles triangles will fulfil the conditions required. For it is evident, from the last Proposition, that isosceles triangles constituted on CB and AB, with each of the angles at the base double the vertical angle, would have AB and AC for their sides, and consequently (I. 2.) must coincide with the triangles now described.

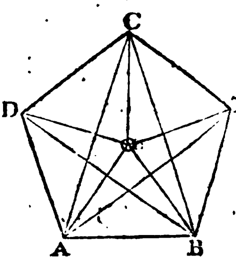
Cor. Hence an isosceles triangle of this kind has its vertical angle equal to the fifth part of two right angles; for each of the angles at the base being double of the vertical angle, they are both equal to four times it, and consequently this vertical angle is the fifth part of all the angles of the triangle, or of two right angles.

PROP. V. PROB.

On a given finite straight line, to describe a regular pentagon.

Let AB be the straight line, on which it is required to describe a regular pentagon.

On AB erect (IV. 4.) the isosceles triangle ACB having each of the angles at the base double of its vertical angle, on AB again construct (IV. 2.) another isosceles triangle whose vertical angle AOB is double of ACB, and about the vertex O place (I. 1.) the isosceles triangles AOD, DOC, COE, and EOB: These triangles, with AOB, will compose a regular pentagon.



For the angle AOB, being the double of ACB, which is the fifth part of two right angles (IV. 4. cor.), must be equal to the fifth part of four right angles; and consequently five angles, each of them equal to AOB, will adapt themselves about the point O. But the bases of those central triangles, and which form the sides of the pentagon, are all equal; and the angles at their bases being likewise equal, they are equal in the collective pairs which constitute the internal angles of the figure: It is therefore a regular pentagon.

Or thus.

Having erected the isosceles triangle ACB, from the centre A with the distance AC describe an arc of a circle, and from the centre B with the same distance describe another arc, and from C inflect the straight lines CE, CD equal to

AB: The points D, E mark out the pentagon. For it is apparent, that, the three straight lines AO, BO, and CO being equal (IV. 2.), and the triangles ACB, CAE, and CBD being likewise equal, the point O must have the same relation to all of them, and consequently the central triangles COD and COE are equal to AOB.

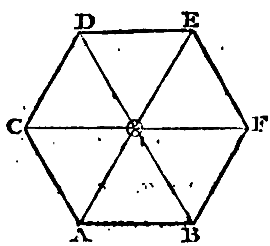
PROP. VI. PROB.

On a given finite straight line, to describe a regular hexagon.

Let AB be the given straight line, on which it is required to describe a regular hexagon.

On AB construct (I. 1.) the equilateral triangle AOB, and repeat equal triangles about the vertex O; these triangles will together compose the hexagon required.

Because AOB is an equilateral triangle, each of its angles is equal to the third part of two right angles (I. 32. cor. 1.); wherefore the vertical angle AOB is the sixth part of four right angles, or six of such angles may be placed about the point O. But the bases of the triangles AOB, AOC, COD, DOE, EOF, and BOF are all equal; and so are the angles at the bases, and which, taken by pairs, form the internal angles of the figure BACDEF. This figure is, therefore, a regular hexagon.

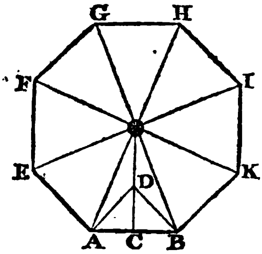


PROP. VII. PROB.

On a given finite straight line, to describe a regular octagon.

Let AB be the given straight line, on which it is required to describe a regular octagon.

Bisect AB (I. 5.) by the perpendicular CD, which make equal to CA or CB, join DA and DB, produce CD until DO be equal to DA or DB; draw AO and BO, thus forming (IV. 1.) an angle equal to the half of ADB, and, about the vertex O, repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and KOB to compose the octagon.



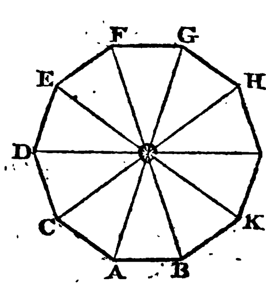
For the distances AD, BD are evidently equal; and because CA, CD, and CB are all equal the angle ADB is contained in a semicircle, and is therefore a right angle (III. 17.). Consequently AOB is equal to the half of a right angle, and eight such angles will adapt themselves about the point O. Whence the figure BAEFGHIK, having eight equal sides and equal angles, is a regular octagon.

PROP. VIII. PROB.

On a given finite straight line, to describe a regular decagon.

Let AB be the straight line, on which it is required to describe a regular decagon.

On AB construct (IV. 4.) an isosceles triangle having each of the angles at its base double of the vertical angle, and, about the point O, place a series of triangles all equal to AOB: A regular decagon will result from this composition.



For the vertical angle AOB of the isosceles triangle is equal to the fifth part of two right angles (IV. 4. cor.), or to the tenth part of four right angles; whence ten such angles may be formed

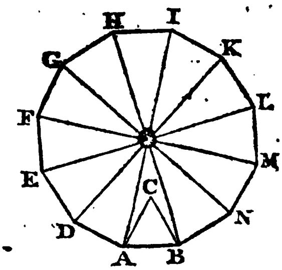
about the point O. The figure BACDEFGHIK, having therefore ten equal sides and equal angles, is a regular decagon.

PROP. IX. PROB.

On a given finite straight line, to describe a regular dodecagon.

Let AB be the straight line, on which it is required to describe a regular twelve-sided figure.

On AB construct (I. 1.) the equilateral triangle ACB, and again (IV. 1.) the isosceles triangle AOB, having its vertical angle equal to the half of ACB, and repeat this triangle AOB about the point O; a regular dodecagon will be thus formed.



For ACB being an equilateral triangle, each of its angles is the third part of two right angles (I. 32. cor. 1.); consequently the angle AOB is the sixth part of two right angles or the twelfth part of four right angles, and twelve such angles can, therefore, be placed about the vertex O.

Scholium. Hence a regular twenty-sided figure may be described on a given straight line, by first constructing on it an isosceles triangle having each of the angles at the base double of the vertical angle, and then erecting another isosceles triangle with its vertical angle equal to the half of this. And, by thus changing the elementary triangle, a regular polygon may be always described, with twice the number of sides.

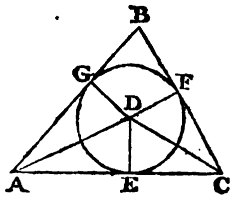
PROP. X. PROB.

In a given triangle, to inscribe a circle.

Let ABC be a triangle, in which it is required to inscribe a circle.

Draw AD and CD (I. 5.) to bisect the angles CAB and ACB, and from their point of concourse D, with its distance DE from the base, describe the circle EFG: This circle will touch the triangle internally.

For let fall the perpendiculars DG and DF upon the sides AB and BC (I. 6.). The triangles ADE, ADG, having the angle DAE equal to DAG, the right angle DEA equal to DGA, and the interjacent side AD common, are equal (I. 21.), and therefore the side DE is equal to DG. In the same manner, it is proved, from the equality of the triangles CDE, CDF, that DE is equal to DF; consequently DG is equal to DF, and the circle passes through the three points E, G, and F. But it also touches (III. 24.) the sides of the triangle in those points, for the angles DEA, DGA, and DFC are all of them right angles.

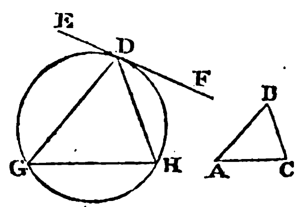


PROP. XI. PROB.

In a given circle, to inscribe a triangle equiangular to a given triangle.

Let GDH be a circle, in which it is required to inscribe a triangle that shall have its angles equal to those of the triangle ABC.

Assuming any point D in the circumference of the circle, draw (III. 24.) the tangent EDF, and make the angles EDG, FDH equal to BCA, BAC, and join GH: The triangle GDH is equiangular to ABC.



For EF being a tangent, and DG drawn from the point of contact, the angle EDG, which

was made equal to BCA, is equal to the angle DHG in the alternate segment (III. 25.); consequently DHG is equal to BCA. And for the same reason, the angle DGH is equal to BAC; wherefore (I. 32.) the remaining angle GDH of the triangle GHD is equal to the remaining angle ABC of the triangle ACB, and these triangles are mutually equiangular.

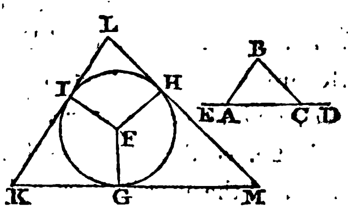
PROP. XII. PROB.

About a given circle, to describe a triangle equiangular to a given triangle.

Let GIH be a circle, about which it is required to describe a triangle, having its angles equal to those of the triangle ABC.

Draw any radius FG, and with it make (I. 4.) the angles GFI, GFH equal to the exterior angles BAE, BCD of the triangle ABC, and, from the points G, I, and H draw the tangents KM, KL, and LM to form the triangle KLM: This triangle is equiangular to ABC.

For all the angles of the quadrilateral figure KIFG being equal to four right angles, and the angles KIF and KGF being each a right angle (III. 24.), the remaining angles GKI and GFI are together equal to two



right angles, and are consequently equal to the angles BAC and BAE on the same side of the straight line ED. But the angle GFI was made equal to BAE; whence GKI is equal to CAB. In like manner, it is proved that the angle GMH is equal to ACB; and the angles at K and M being thus equal to BAC and BCA, the remaining angle at L is (I. 32.) equal to that at B, and the two triangles are, therefore, equiangular.

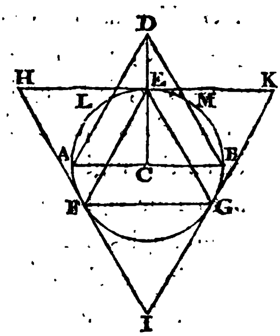
PROP. XIII. PROB.

In and about a given circle, to inscribe and circumscribe an equilateral triangle.

Let AEB be a circle, in which it is required to inscribe an isosceles triangle.

Draw the diameter AB, describe (I. 1.) the equilateral triangle ADB, join CD meeting the circumference in E, draw (I. 24.) EF, EG parallel to AD, BD, and join FG: The triangle EFG is equilateral.

For the triangles ADC, BDC having the two sides DA, AC equal to DB, BC, and the third side DC common to both, are (I. 2.) equal, and the angle DCA is equal to DCB; whence the arc AE is (III. 14.) equal to BE. And the triangle ADB (I. 11. cor.) being likewise equiangular, the angle DBA is equal to DAB, and the arc AEM equal to BEL, and the remaining arc ME equal to LE. But EF and EG being parallel to LA and MB, the arcs AF and BG are (III. 20.) equal to LE and ME, and to each other; hence (III. 20. cor.) FG is parallel to AB, and the inscribed triangle FEG is (I. 31.) equiangular, and consequently equilateral.



Again, let it be required to describe an equilateral triangle about the circle AEB.

The same construction remaining; at the points F, E, and G, apply the tangents HI, HK, and KI, to form the circumscribing triangle IHK: This triangle is equilateral.

For because IH is a tangent and FG is inflected from the point of contact, the angle IFG is equal to the angle FEG in the alternate segment (III. 25.), and therefore IH is parallel

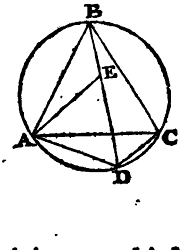
to EG (I. 23. cor.). In like manner it is proved, that HK, KI are parallel to GF, FE, and consequently (I. 31.) the angles of the triangle IHK are equal to those of FEG, and therefore equal to each other.

Cor. Hence the circumscribing equilateral triangle contains four times that which is inscribed; for the figures EFIG, EHFG, and EFGK are evidently equal rhombuses, and contain equilateral triangles which are all equal. Hence also the side of the circumscribing, is double of that of the inscribed, equilateral triangle.

PROP. XIV. THEOR.

A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two chords inflected from the same point to the extremities of the base.

Let ABC be an equilateral triangle inscribed in a circle, and BD, AD, and CD chords drawn from it to a point D in the circumference; BD is equal to AD and CD taken together.



For, make DE equal to DA, and join AE. The angle ADB, being (III. 18.) equal to ACB in the same segment is equal (I. 32. cor.) to the third part of two right angles. But the triangle ADE being isosceles by construction, the angles DAE, DEA at its base are equal (I. 11.), and each of them is, therefore, equal to half of the remaining two-thirds of two right angles, or to one-third part. Consequently ADE is an equilateral triangle (I. 12. cor.), and the angle DAE equal to CAB; take CAE from both, and there remains the angle DAC equal to EAB; but the angle ABD is equal to

ACD in the same segment. And thus the triangles ADC and AEB have the angles DAC, DCA equal to EAB, EBA, and the interjacent side AC equal to AB; they are consequently equal (I. 21.), and the side DC is equal to EB. But DE was made equal to DA; wherefore DA and DC are together equal to DE and EB, or to DB.

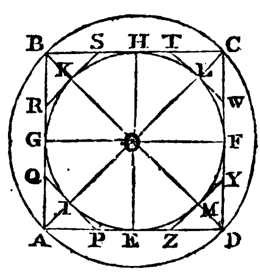
PROP. XV. PROB.

About and in a given square, to circumscribe and inscribe a circle.

Let ABCD be a figure, about which it is required to circumscribe a circle.

Draw the diagonals AC, DB, intersecting each other in O, and, from that point with the distance AO, describe the circle ABCD: This circle will circumscribe the square.

Because the diagonals of the square ABCD are equal and bisect each other (I. 29. and cor.), the straight lines OA, OB, OC, and OD are all equal, and consequently the circle described through A passes through the other points B, C, and D.



Again, let it be required to inscribe a circle in the square ABCD.

From O the intersection of the diagonals and with its distance from the side AD, describe the circle EGHF: This circle will touch the square internally.

For let fall the perpendiculars OG, OH, and OF (I. 6.). And because the straight lines AB, BC, CD, and DA are equal, they are equally distant from the centre O of the exterior circle (III. 11.); wherefore the perpendiculars OE, OG, OH, and OF are all equal, and the interior circle passes

through the points G, H, and F; but (I. 24.) it likewise touches the sides of the square, since they are perpendicular to the radii drawn from O.

Cor. Hence an octagon may be inscribed within a square. For let tangents be applied at the points I, K, L, and M, where the diagonals cut the interior circle. It is evident, that the triangle AOE is equal to DOE, IOP to EOP, and EOZ to MOZ; whence the angles POE and ZOE are equal, being the halves of EOA and EOD, and consequently the triangles PEO and ZEO are equal. Wherefore PZ, the double of PE, is equal to PQ, the double of PI; and the angle EZM is, for a like reason, equal to EPI. And, in this manner, all the sides and all the angles about the eight-sided figure PQRSTWYZ are proved to be equal.

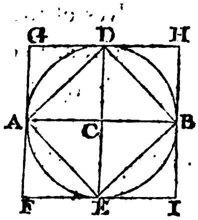
PROP. XVI. PROB.

In and about a given circle, to inscribe and circumscribe a square.

Let EADB be a circle in which it is required to inscribe a square.

Draw the diameter AB, the perpendicular ED, and join AD, DB, BE, and EA: The inscribed figure ADBE is a square.

The angles about the centre C, being right angles, are equal to each other, and are, therefore, subtended by equal chords AD, DB, BE, and AE, but one of the angles ADB, being in a semicircle, is (I. 22.) a right angle, and consequently ADBE is a square.



Next, let it be required to circumscribe a square about the circle.

Apply tangents FG, GH, HI, and FI at the extremities of the perpendicular diameters: These will form a square.

For all the angles of the quadrilateral figure CG being together equal to four right angles, and those at C, A, and D being each a right angle, the remaining angle at G is also a right angle, CG is a rectangle; and AC being equal to CD, it is likewise a square. In the same manner, CH, CI, and CF are proved to be squares; the sides FG, GH, HI, and IF of the exterior figure, being therefore the doubles of equal lines, are mutually equal, and the angle at G being a right angle, FH is consequently a square.

Cor. Hence the circumscribing square is double of the inscribed square, and this again is double of the square described on the radius of the circle.

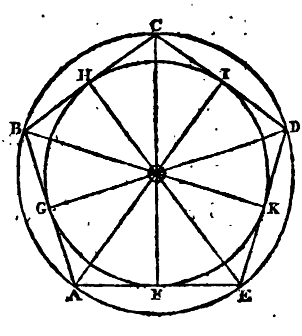
PROP. XVII. PROB.

To inscribe and circumscribe a circle in and about a given regular pentagon.

Let ABCDE be a regular pentagon, in which it is required to inscribe a circle.

Draw AO and EO to bisect the angles at A and E, join C with the point of concourse O and produce it to meet AE in F, and from O as a centre, with the distance OF, describe a circle FGHIK: This circle will touch the pentagon internally.

For, from the point O, let fall perpendiculars on the opposite sides of the figure. The angles EAO and AEO, being the halves of the angles of the pentagon, are equal, and consequently the triangle AOE is isosceles, and the perpendicular OF bisects the base. And the triangles AOG and BOG, having the angles OAG and OGA equal to OBG and OGB and the common side OG, are (I. 3.) equal. Again the tri-



angles BOG and BOH have now the angles OBG and OGB equal to OBH and OHB, with the side BO common to both, and are therefore equal. In like manner, all the triangles about the centre O are proved to be equal; consequently the perpendiculars OF, OG, OH, OI, and OK are equal, and the circle touches the pentagon in the points F, G, H, I, and K.

Next, let it be required to describe a circle about the pentagon.

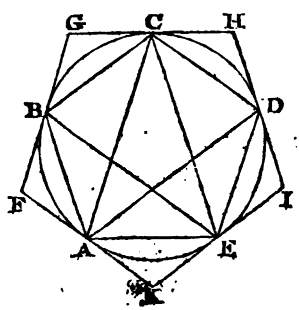
From the same centre O, with the distance OA, describe a circle: It will pass through the points B, C, D, E; for the triangles about O being all equal, the straight lines OA, OB, OC, OD, and OE must be likewise equal.

PROP. XVIII. PROB.

In and about a given circle, to inscribe and circumscribe a regular pentagon.

Let ABCDE be a circle in which it is required to inscribe a regular pentagon.

Construct an isosceles triangle having each of its angles at the base double of its vertical angle (IV. 4.), and equiangular to this, inscribe the triangle ACE within the circle (IV. 11.), draw AD, EB bisecting the angles CAE, CEA (I. 5.), and join AB, BC, CD, and DE: The figure ABCDE is a regular pentagon.



For the angles AEB, BEC are each the half of CEA, and therefore equal to ACE; but the angles EAD, DAC are likewise equal to ACE. Hence these angles, being all equal, must stand on equal arcs (III. 18. cor.); and the chords of these arcs, or the sides AB, BC, CD, DE, and AE are equal (III. 13. cor.). And because the segments EAB, ABC, BCD, CDE, and

DEA are evidently equal, the interior angles of the figure are all equal (III. 18.), and it is, therefore, a regular pentagon.

Next, let it be required to circumscribe a regular pentagon about the circle.

At the points A, B, C, D, and E apply tangents; these will form a regular pentagon.

For FAK being a tangent, the angle KAE is equal to ACE (III. 25.); and in like manner it is shown that the angles AEK, DEI, EDI, CDH, DCH, BCG, CBG, ABF, BAF are all equal to ACE. The isosceles triangles AKE, BFA, having, therefore, the angles at the base equal and the bases themselves AE, AB,—are equal (I. 21.); for the same reason, the triangles BGC, CHD, DIE, EKA, are equal. Whence the internal angles of the figure are equal, and its sides, being double of those of the annexed triangles, are likewise equal: The figure is, therefore, a regular pentagon.

PROP. XIX. PROB.

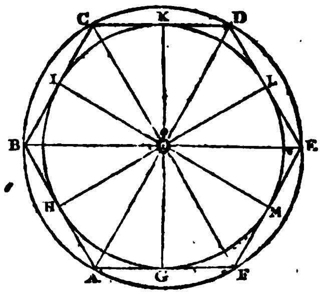
In and about a regular hexagon, to inscribe and circumscribe a circle.

Let ABCDEF be a regular hexagon, in which it is required to inscribe a circle.

Draw AO and FO, bisecting the angles BAF and AFE (I. 5.); and from the point of intersection O, with its distance from the side AF, describe a circle: This circle will touch the hexagon internally.

For let fall perpendiculars from O upon the sides of the figure. It may be demonstrated, as in Prop. XVII. that the triangles AOB, BOC, COD, DOE, and EOF are all equal

to AOF; and, in like manner, it will appear that the intermediate bisected triangles are equal. Hence the perpendiculars OG, OH, OI, OK, OL, and OM, are all equal, and a circle must touch these at the points G, H, I, K, L, and M.



Again, let it be required to describe a circle about the hexagon.

From the same point O, as a centre, with the distance OA, describe a circle, which must pass through the points B, C, D, E, and F; for the straight lines OA, OB, OC, OD, OE, and OF were proved to be equal.

Cor. Hence, in any regular polygon, the centre of the inscribing and circumscribing circle is the same, and may be determined in general, by drawing lines to bisect the adjacent angles of the figure.

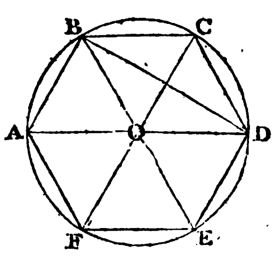
PROP. XX. PROB.

To inscribe a regular hexagon in a given circle.

Let it be required, in the circle FBD, to inscribe a hexagon.

Draw the radius OA, on which construct the equilateral triangle ABO (I. 1. cor.), and repeat the equal triangles about the vertex O: These triangles will compose a hexagon.

For the triangle ABO, being equilateral, each of its angles, AOB, is the third part of two right angles; and consequently six of such angles may be placed about the centre O. But the bases of the triangles AOB, BOC, COD, DOE, and EOF form the sides of the figure, and the angles at those bases its internal angles; wherefore it is a regular hexagon.



Cor. 1. Tangents applied at the points A, B, C, D, E, and

F, would evidently form a regular circumscribing hexagon.— An equilateral triangle might be inscribed by joining the alternate points; and, by applying tangents at those points, an equilateral triangle would be made to circumscribe the circle.

Cor. 2. The side AB of the inscribed hexagon is equal to the radius; and since ABD is a right-angled triangle, and the squares of AB and BD are equal to the square of AD or to four times the square of AO, the square of BD the side of an inscribed equilateral triangle is triple the square of the radius.

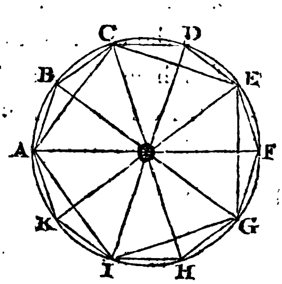
Cor. 3. The perimeter of the inscribed hexagon is equal to six times the radius, or three times the diameter, of the circle. Hence the circumference of a circle being, from its perpetual curvature, greater than any intermediate system of straight lines, is more than triple its diameter.

PROP. XXI. PROB.

To inscribe a regular decagon in a given circle.

Let ADH be a circle, in which it is required to inscribe a regular decagon.

Draw the radius OA, and with OA as its side describe the isosceles triangle AOB, having each of its angles at the base double of its vertical angle (IV. 4.), repeat the equal triangles about the centre O: These triangles will compose a decagon.



For the vertical angle AOB of the component isosceles triangle, is the fifth part of two right angles (IV. 4. cor.), and consequently ten such angles can be placed about the point O. But the sides and angles of the resulting figure are all evidently equal; it is, therefore, a regular decagon.

Cor. Hence a regular pentagon will be formed, by joining

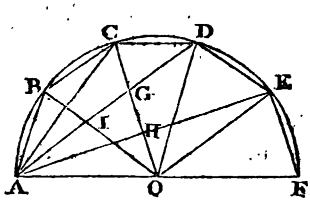
the alternate points A, C, E, G, I, and A. It is also manifest, that a decagon and a pentagon may be circumscribed about the circle, by applying tangents at their several angular points.

PROP. XXII. THEOR.

The square of the side of a pentagon inscribed in a circle, is equivalent to the squares of the sides of the inscribed hexagon and decagon.

Let ABCDEF be half of a decagon inscribed in a circle, whose diameter is AF; the square of AC the side of the incribed pentagon, is equivalent to the square of AB the side of the inscribed decagon, and the square of the radius AO which is equal to a side of the inscribed hexagon.

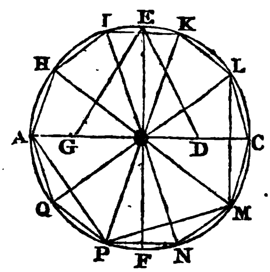
For join AD, AE, and draw OB, OC, OD, and OE. The angle FAD at the circumference, being half of the angle FOD at the centre (III. 17.), is equal to the angle AOB; and, for the same reason, the angle FAB, being half of FOB, is equal to FOD or COA. The triangles ABO and AGO, having, therefore, the angles AOB, OAB equal to OAG, AOG, and the side AO common to both, are equal (I. 21.) and isosceles, and consequently the base AB is equal to OG. But the angles FAE and EAD, standing on equal arcs, are equal (III. 18. cor.); wherefore the triangles OAH and GAH, having the side AG equal to AO, the side AH common, and the contained angle OAH equal to GAH, are equal (I. 3.), and hence OH is equal to GH, and the angles AHO and AHG are equal and right angles. And because AO is equal to CO and AH perpendicular to it, the square of AC is equivalent to twice the rectangle under OC and CH (II. 26. cor.), or the rectangle under OC and twice CH, which is evidently the



sum of OC and CG. The square of AC is, therefore, equivalent to the square of OC, with the rectangle under OC and CG; but OG being equal to AB, the radius OC is divided by a medial section at G, and consequently the rectangle under OC and CG is equivalent to the square of OG or AB. Whence the square of AC is equivalent to the two squares of AO and AB.

Cor. 1. The triple chord AD of the decagon, is equal to the sides AO and AB of the inscribed hexagon and decagon. For AO being equal to DO, the angle OAD is equal to ODA (I. 11.); but OAD, or FAD, is equal to the angle DOC (III. 17.), and consequently the angle DOG is equal to ODG, and the side OG equal to DG (I. 12.) Wherefore AD being equal to AG and GD, is equal to AO with OG or AB.

Cor. 2. Hence the sides of the inscribed decagon and pentagon may be found by a single construction. For draw the perpendicular diameters AC and EF, bisect OC in D, join DE, make DG equal to it, and join GE. It is evident, that AO is cut medially in G (II. 22.), and consequently that OG is equal to a side of the inscribed decagon. But GOE being a right-angled triangle,



the square of GE is equivalent to the squares of GO and OE (II. 11.), or the squares of the sides of the decagon and hexagon; whence GE is equal to the side of the inscribed pentagon. It also follows, that CG is equal to CI or CP, the triple chords of the inscribed decagon *.

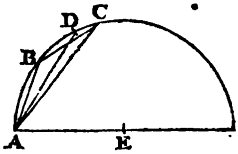
PROP. XXIII. PROB.

In a given circle, to inscribe regular polygons of fifteen and of thirty sides.

---

* See Note XXXI.

Let AB and BC be the sides of an inscribed decagon, and AD the side of a hexagon inscribed; the arc BD will be the fifteenth part of the circumference of the circle, and DC the thirtieth part.



For, if the circumference were divided into thirty equal portions, the arc AB would be equal to three of these, and the arc AD to five; consequently the excess BD is equal to two of these portions, or it is the fifteenth part of the whole circumference. Again, the double arc ABC being equal to six portions, and ABD to five, the defect DC is equal to one portion, or to the thirtieth part of the circumference.

Scholium. From the inscription of the square, the pentagon, and the hexagon,—may be derived that of a variety of other regular polygons: For, by continually bisecting the intercepted arcs and inserting new chords, the inscribed figure will, at each successive operation, have the number of its sides doubled. Hence polygons will arise of 6, 8, and 10 sides; then, of 12, 16, and 20; next of 24, 32, and 40; again, of 48, 64, and 80; and so forth repeatedly. The excess of the arc of the hexagon and above that of the decagon, gives the arc of a fifteen-sided figure; and the continued bisection of this arc will mark out polygons with 30, 60, or 120 equal sides, in perpetual succession. The same results might also be obtained from the differences of the preceding arcs *.

Of the regular polygons, three only are susceptible of perfect adaptation, and capable therefore of covering, by their repeated addition, a plane surface. These are the equilateral triangle, the square, and the hexagon. The angles of an equilateral triangle are each two-thirds of a right angle, those of a square are right angles, and the angles of a hexagon are

* See Note XXXII.

each equal to four-third parts of a right angle. Hence there may be constituted about a point, six equilateral triangles, four squares, and three hexagons. But no other regular polygon can admit of a like disposition. The pentagon, for instance, having each of its angles equal to six-fifths of a right angle, would not fill up the whole space about a point, on being repeated three times; yet it would do more than cover that space, if added four times. On the other hand, since each angle of a polygon which has more than six sides must exceed four third parts of a right angle, three such polygons cannot stand round a point. Nor can the space about a point ever be bisected by the application of any regular polygons, of whatever number of sides; for their angles are always necessarily each less than two right angles *.

---

* See Note XXXIII.

ELEMENTS

OF

GEOMETRY.

BOOK V.

OF PROPORTION.

THE preceding Books treat of magnitude as *concrete*, or having mere extension; and the simpler properties of lines, of angles, and of surfaces, were deduced, by a continuous process of reasoning, grounded originally on superposition. But this mode of investigation, however satisfactory to the mind, is, from its nature, very limited and laborious. By introducing the idea of *Number* into geometry, a new scene is opened, and a far wider prospect rises into view. Magnitude, being considered as *discrete*, or composed of integrant parts, becomes assimilated to *multitude;* and under that aspect, it presents a vast system of relations, which may be traced out with the utmost facility.

Numbers were first employed, to denote the collection of distinct, though kindred, objects; but the subdivision of *extent*, whether actually effected or only conceived to exist, bestowing a sort of individuality, they came afterwards to acquire a more comprehensive application. In comparing together two quantities of the same kind, the one may *contain* the other, or be *contained* by it; that is, the one may result from the repeated addition of the other, or it may in its turn produce this other by a successive composition. The one quantity is, therefore, equal, either to so many times the other, or to a certain aliquot part of it.

Such seems to be the simplest of numerical relations. It is very confined, however, in its application, and is evidently, in that shape, insufficient altogether for the purpose of general comparison. But this object is attained, by adopting some intermediate reference. Though a quantity neither contain another exactly, nor be contained by it; there may yet exist a third and smaller quantity, which is at once capable of *measuring* them both. This *measure* corresponds to the arithmetical unit; and as *number* denotes the collection of units, so *quantity* may be viewed as the aggregate of its component measures.

But mathematical quantities are not all susceptible of such perfect mensuration. Two quantities may be *conceived* to be so constituted, as not to admit of any other that will measure them completely, or be contained in both without leaving a remainder. Yet this apparent imperfection, which proceeds entirely from the infinite variety ascribed to

possible magnitude, creates no real obstacle to the progress of accurate science. The measure or primary element, being assumed successively still smaller and smaller, its corresponding remainder must be perpetually diminished. This continued exhaustion will hence approach to its absolute term, nearer than by any assignable difference.

Quantities in general can, therefore, either exactly or to any required degree of precision, be represented abstractly by numbers; and thus the science of Geometry is at last brought under the dominion of Arithmetic.

It is obvious, that quantities of any kind must have the same composition, when each contains its measure the same number of times. But quantities, viewed in pairs, may be considered as having a similar composition, if the corresponding terms of each pair contain its measure equally. Two pairs of quantities of a similar composition, being thus formed by the same distinct aggregations of their elementary parts, constitute a *proportion* *.

* See Note XXXIV.

DEFINITIONS.

1. Quantities are *homogeneous*, which can be added together.

2. One quantity is said to *contain* another, when the subtraction of this—continued if necessary—leaves no remainder.

3. A quantity which is contained in another, is said to *measure* it.

4. The quantity which is measured by another, is called its *multiple*; and that which measures the other, its *submultiple*.

5. *Like* multiples and submultiples are those which contain their measures equally, or which equally measure their corresponding compounds.

6. Quantities are *commensurable*, which have a finite common measure; they are *incommensurable*, if they will admit of no such measure.

7. That relation which one quantity is conceived to bear to another in regard to their composition, is named a *ratio*.

8. When both terms of comparison are equal, it is called a ratio of *equality*; if the first of these be greater than the second, it is a ratio of *majority*; and if the first be less than the second, it is a ratio of *minority*.

9. A *proportion* or *analogy* consists in the identity of ratios.

10. Four quantities are said to be *proportional*, when a submultiple of the first is contained in the second as often as a like submultiple of the third is contained in the fourth.

11. Of proportional quantities, the first of each pair is named the *antecedent*, and the second the *consequent*.

12. The antecedents are *homologous* terms; and so are the consequents.

13. One antecedent is said *to be* to its consequent, *as* another antecedent *to* its consequent.

14. The first and last terms of a proportion are called the *extremes*, and the intermediate ones, the *means*.

15. A ratio is *direct*, if it follows the order of the terms compared; it is *inverse* or *reciprocal*, when it holds a reversed order.

Thus, if the ratio of A to B be *direct*, that of B to A is the *inverse* or *reciprocal* ratio.

16. Quantities form a *continued proportion*, when the intervening terms stand in the double relation of consequents and antecedents.

17. When a proportion consists of three terms, the middle one is said to be a *mean proportional* between the two extremes.

18. The ratio which one quantity has to another may be considered as *compounded* of all the connecting ratios among any interposed quantities.

Thus, the ratio of A to D is viewed as *compounded* of that of A to B, that of B to C, and that of C to D.

19. Of quantities in a continued proportion, the first is

said to have to the third, a ratio the *duplicate* of what it has to the second; to have to the fourth, a *triplicate* ratio; to the fifth, a *quadruplicate* ratio; and so forth, according to the number of equal ratios inserted between the extreme terms.

20. If quantities be continually proportional, the ratio of the first to the second is called the *subduplicate* of the ratio of the first to the third, the *subtriplicate* of the ratio of the first to the fourth, &c.

21. A straight line is said to be cut in *extreme and mean ratio*, when the one segment is a mean proportional between the other segment and the whole line.

To facilitate the language of demonstration relative to numbers or abstract quantities, it is expedient to adopt a clear and concise mode of notation.

1. The sign $=$ expresses *equality*, $>$ *majority*, and $<$ *minority*: Thus $A=B$ denotes that A is equal to B, $A>B$ signifies that A is greater than B, and $A<B$ imports that A is less than B.

2. The signs $+$ and $-$ mark the addition and subtraction of the quantities to which they are prefixed: Thus, $A+B$ denotes that B is to be joined to A, and $A-B$ signifies that B is to be taken away from A. Sometimes these two symbols are combined together: Thus, $A\pm B$ represents either the sum of A and B, or the excess of A above B.

3. To express multiplication, the quantities are placed close together; or they may be connected by the point (.), or the cross $\times$: Thus, AB, or A.B, or $A\times B$, denotes the product of A by B; and ABC indicates the result of the continued multiplication of A by B, and of this product again by C.

4. When the same number is repeatedly multiplied, the product is termed its *power*; and the number itself, in reference to that power, is called the *root*. The notation is here still farther abridged, by retaining only a single letter with a small figure over it, to mark how often it is understood to be repeated: This figure serves also to distinguish the order of the power. Thus AA, or A^2, signifies that A is multiplied by A, and that the product is the *second power* of A; and AAA, or A^3, in like manner, imports that AA is again multiplied by A, and that the result is the *third power* of A.

5. The roots are denoted, by prefixing a contracted *r*, or the symbol $\surd$. Thus $\surd A$ or $\sqrt[2]{A}$ marks the *second root* of A, or that number of which A is the second power; $\sqrt[3]{A}$ signifies the *third root* of A, or the number which has A for its third power.

6. To represent the multiplication of complex quantities, they are included by a parenthesis. Thus, $A(B+C-D)$ denotes that the amount of $B+C-D$, considered as a single quantity, is multiplied into A.

7. Ratios and analogies are expressed, by inserting points in pairs between the terms. Thus $A : B$ denotes the ratio of A to B, and the compound symbols $A : B :: C : D$, signify that the ratio of A to B is the same as that of C to D, or that A is to B as C to D.

PROP. I. THEOR.

The product of a number into the sum or difference of two numbers, is equal to the sum or difference of its products by those numbers.

Let A, B, and C be three numbers; the product of the sum or difference of B and C by the number A, is equal to the sum or difference of the products AB and AC.

For the product AB is the same as each unit contained in B repeated A times, and the product AC is the same as the units in C likewise repeated A times; whence the sum of the products AB and AC, is equal to the units contained in both B and C, all repeated A times, or it is equal to the sum of the numbers B and C multiplied by A.

Again, for the same reason, the difference between the products AB and AC must be equal to the difference between the units contained in B and in C, repeated A times; that is, it must be equal to the difference between the numbers B and C multiplied by A.

Cor. 1. Hence a number which measures any two numbers, will measure also their sum and their difference.

Cor. 2. It is hence manifest, that the first part of the proposition may be extended to more numbers than two; or that $AB+AC+AD+$, &c.$=A(B+C+D+$, &c.)

PROP. II. THEOR.

The product which arises from the continued multiplication of any numbers, is the same, in whatever order that operation be performed.

Let A and B be two numbers; the product AB is equal to BA.

For the product AB is the same as each unit in B added

together A times, that is, the same as A itself repeated B times, or BA.

Next, let there be three numbers A, B, and C; the products ABC, ACB, BAC, BCA, CAB, and CBA are all equal.

For put $D=AB$ or BA; then $DC=CD$, that is, $ABC=CAB$, and $BAC=CBA$.

Again, put $E=AC$ or CA; then $EB=BE$, that is, $ACB=BAC$, and $CAB=BCA$.

Lastly, put $F=BC$ or CB; then $FA=AF$, that is, $BCA=ABC$, and $CBA=ACB$.

And thus the several products are all mutually equal.

It is also manifest, that the same mode of reasoning might be extended to the products of any multitude of numbers.

PROP. III. THEOR.

Homogeneous quantities are proportional to their like multiples or submultiples.

Let A, B be two quantities of the same kind, and pA, pB their like multiples; $A : B :: pA : pB$.

For, since A and B are capable of being measured to any required degree of precision, suppose $A=m.a$ and $B=n.a$; then $pA=p.ma$, and $pB=p.na$. But (V. 2.) $p.ma=m.pa$, and $p.na=n.pa$. Wherefore a and pa are like submultiples of A and of pA, which contain them respectively m times; and these like submultiples are both contained equally, or n times, in B and in pB. Consequently (V. def. 10.) the quantities A, B, and pA, pB are proportional; and A, pA are the antecedents, and B, pB, the consequents, of the analogy.

Again, because the ratio of pA to pB is thus the same as that of A to B, which, in reference to pA and pB, are only like submultiples, it follows that homogeneous quantities are also proportional to their like submultiples.

PROP. IV. THEOR.

In proportional quantities, according as the first term is greater, equal, or less than the second, the third term is greater, equal, or less than the fourth.

Let $A : B :: C : D$; then if $A > B$, $C > D$; if $A = B$, $C = D$; and if $A < B$, $C < D$.

For, if A be greater than B, $A : B$ is a ratio of majority; whence $C : D$, being the same with it, is likewise a ratio of majority, and consequently C is greater than D.

If A be equal to B, $A : B$ must be a ratio of equality, and hence $C : D$ is also a ratio of equality, or C is equal to D.

But, if A be less than B, $A : B$ is a ratio of minority, and so is, therefore, $C : D$, or C is less than D.

PROP. V. THEOR.

Of four proportionals, if the first be a multiple or submultiple of the second, the third is a like multiple or submultiple of the fourth.

Let $A : B :: C : D$; if $A = pB$, then $C = pD$.

For, suppose the approximate measures of A and C to be a and c, and let $A = mp.a$, and $C = mp.c$. It is evident, from the hypothesis, that, $A = pB = mp.a$, or $B = m.a$; but the consequents B and D must contain their measures equally (V. def. 10.), and therefore $D = m.c$. Whence $C = mp.c =$ (V. 2.) $p.mc = pD$.

Again, if $qA = B$; then will $qC = D$.

For, let $A = na$, and $C = nc$; therefore $B = qA = qna =$ (V. 2.) $nq.a$, and, from the definition of proportion, $D = nq.c =$ (V. 2.) $q.nc = qC$.

PROP. VI. THEOR.

If four numbers be proportional, the product of the extremes is equal to that of the means; and of two equal products, the factors are convertible into an analogy, of which these form severally the extreme and the mean terms.

Let A : B : : C : D; then AD=BC.

For (V. 3.) A.D : B.D : : B.C : B.D; and the second term of this analogy being equal to the fourth, therefore (V. 4.) AD=BC.

Again, let AD=BC; then A : B : : C : D.

For, by identity of ratios, AD : BD : : BC : BD, and hence (V. 3.) A : B : : C : D.

Cor. 1. Hence the greatest and least terms of a proportion are either extremes or means.

Cor. 2. Hence also a proportion is not affected by transposing or interchanging its extreme and mean terms.—On this principle, is founded the two following theorems.

PROP. VII. THEOR.

The terms of an analogy are proportional by *inversion*, or the second is to the first, as the fourth to the third.

Let A : B : : C : D; then *inversely* B : A : : D : C.

For the extreme and mean terms are thus only mutually interchanged, and consequently the same equality of products AD and BC still obtains.

PROP. VIII. THEOR.

Numbers are likewise proportional by *alternation*; or the first is to the third, as the second to the fourth.

Let A : B : : C : D; then *alternately* A : C : : B : D.

For the extreme terms are still retained, and the mean terms are merely transposed with respect to each other; the same equality, therefore, of products here also subsists.

PROP. IX. THEOR.

The terms of an analogy are proportional by *composition;* or the sum of the first and second is to the second, as the sum of the third and fourth to the fourth.

Let A : B : : C : D; then by *composition* A+B : B : : C+D : D.

Because A : B : : C : D, the product AD=BC (V. 6.); add to each of these the product BD, and AD+BD=BC+BD. But (V. 1.) AD+BD=D(A+B), and BC+BD=B(C+D); wherefore (V. 6.) assuming the factors of these equal products for the extreme and mean terms, A+B : B : : C+D : D.

PROP. X. THEOR.

The terms of an analogy are proportional by *division;* or the difference of the first and second is to the second, as the difference of the third and fourth to the fourth.

Let A : B : : C : D; suppose A to be greater than B, then will C be greater than D (V. 4.): It is to be proved that A—B : B : : C—D : D.

For, since A : B : : C : D, the product AD=BC (V. 6.), and, taking BD from both, the compound product AD—BD is equal to BC—BD; wherefore, by resolution, (A—B) D=B(C—D), and consequently A—B : B : : C—D : D.

If B be greater than A, then $BD—AD=BD—BC$, and, by resolution, $(B—A)\,D=B\,(D—C)$; whence $B—A : B :: D—C : D$.

PROP. XI. THEOR.

The terms of an analogy are proportional by *conversion;* that is, the first is to the sum or difference of the first and second, as the third to the sum or difference of the third and fourth.

Let $A : B :: C : D$, and suppose $A > B$; then $A : A \pm B :: C : C \pm D$.

For, since (V. 6.) the product $AD=BC$, add or subtract these to or from the product AC; and $AC \pm AD = AC \pm BC$. Wherefore, by resolution, $A(C \pm D)=C(A \pm B)$, and consequently $A : A \pm B :: C : C \pm D$.

If $A < B$, then $AD—AC=BC—AC$, and, by resolution, $A(D—C)=C(B—A)$, whence $A : B—A :: C : D—C$.

Cor. Hence, by inversion, $A \pm B : A :: C \pm D : C$, or $B—A : A :: D—C : C$.

PROP. XII. THEOR.

The terms of an analogy are proportional by *mixing;* or the sum of the first and second is to the difference, as the sum of the third and fourth to their difference.

Let $A : B :: C : D$, and suppose $A > B$; then $A+B : A—B :: C+D : C—D$.

For, by conversion, $A : A+B :: C : C+D$, and alternately $A : C :: A+B \quad C+D$.

Again, by conversion, $A : A—B :: C : C—D$, and alternately $A : C :: A—B : C—D$. Whence, by identity of ra-

tios, A+B : C+D :: A−B : C−D, and alternately A+B : A−B :: C+D : C−D.

The same reasoning will hold if A be less than B, the order of these terms being only changed.

PROP. XIII. THEOR.

A proportion will subsist, if the homologous terms be multiplied by the same numbers.

Let A : B :: C : D; then pA : qB :: pC : qD.

For, since A : B :: C : D, alternately A : C :: B : D; but the ratio of A to C is the same as pA : pC (V. 3.), and the ratio of B to D is the same as qB : qD. Wherefore pA : pC :: qB : qD, and, by alternation, pA : qB :: pC : qD.

Cor. The Proposition may be extended likewise to the division of homologous terms, by employing submultiples.

PROP. XIV. THEOR.

The greatest and least terms of a proportion, are together greater than the intermediate ones.

Let A : B : : C : D; and A being supposed to be the greatest term, the other extreme D is the least (V. 6. cor. 1.): The sum of A and D is greater than the sum of B and C.

Because A : B :: C : D, by conversion A : A—B :: C : C—D, and alternately A : C :: A—B : C—D; but A, being the greatest term, is therefore greater than C, and consequently (V. 4.) A—B is greater than C—D; to each add B+D, and (A+D)>(B+C.)

The same mode of reasoning is applicable, should any other term of the analogy be supposed to be the greatest.

Cor. Hence the mean term of three proportionals, is less than half the sum of both extremes *.

---

* See Note XXXV.

PROP. XV. THEOR.

If two analogies have the same antecedents, another analogy may be formed, having the consequents of the one as antecedents, and those of the other as consequents.

Let A : B : : C : D and A : E : : C : F; then B : E : : D : F.

For, alternating the first analogy, A : C : : B : D, and alternating the second, A : C : : E : F; whence, by identity of ratios, B : D : : E : F,—which inference is named a *direct equality*.

PROP. XVI. THEOR.

If the consequents of one analogy be antecedents in another, a third analogy will obtain, having the same antecedents as the former, and the same consequents as the latter.

Let A : B : : C : D, and B : E : : D : F; then A : E : : C : F.

For, alternating both analogies, A : C : : B : D, and B : D : : E : F; whence, by identity of ratios, A : C : : E : F,—which conclusion is also named a *direct equality*.

PROP. XVII. THEOR.

If two analogies have the same means, the extremes of the one, with those of the other as mean terms, will form a third analogy.

Let A : B : : C : D, and E : B : : C : F; then A : E : : F : D.

For, since A : B : : C : D, AD=BC (V. 6.); and because E : B : : C : F, EF=BC. Whence AD=EF, and A : E : : F : D.

Cor. Hence the extreme and mean terms being interchangeable, it likewise follows, that, if A : B : : C : D, and A : E : : F : D, then B : E : : F : C.

PROP. XVIII. THEOR.

If the extremes of one analogy are the mean terms in another, a third analogy will subsist, having the means of the former as its extremes, and the extremes of the latter as its means.

Let A : B : : C : D, and E : A : : D : F; then B : E : : F : C.

For, from the first analogy AD=BC, and, from the second, EF=AD; whence BC=EF, and consequently B : E : : F : C.

Cor. Hence also, if A : B : : C : D and B : E : : F : C; then E : A : : D : F. The principle of this and the preceding Proposition, is named *inverse*, or *perturbate*, *equality*.

PROP. XIX. THEOR.

If there be any number of proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let A : B : : C : D : : E : F : : G : H; then A : B : : A+C+E+G : B+D+F+H.

Because A : B : : C : D, AD=BC; and since A : B : : E : F, AF=BE, and, for the same reason, AH=BG. Consequently, the aggregate products, AB+AD+AF+AH=BA+BC+BE+BG, and, by resolution, A(B+D+F+H)=B(A+C+E+G), whence A : B : : A+C+E+G : B+D+F+H.

Cor. 1. It is obvious, that the Proposition will extend likewise to the difference of the homologous terms, and may, therefore, be more generally expressed thus: $A : B :: A \pm C \pm E \pm G : B \pm D \pm F \pm H$.

Cor. 2. Hence in continued proportionals, as one antecedent is to its consequent, so is the sum or difference of the several antecedents to the corresponding sum or difference of the consequents. For, if $A : B :: B : C :: C : D$; then $A : B :: A \pm B \pm C : B \pm C \pm D$; or, omitting B and C which stand in the relation of antecedent and consequent, $A : B$, or $B : C :: A \pm C : B \pm D$.

PROP. XX. THEOR.

If two analogies have the same antecedents, another analogy may be formed of these antecedents, and the sum or difference of the consequents.

Let $A : B :: C : D$, and $A : E :: C : F$; then $A : B \pm E :: C : D \pm F$. For, by alternation, these analogies become $A : C :: B : D$, and $A : C :: E : F$; whence (V. 19.) $A : C :: B \pm E : D \pm F$, and alternately $A : B \pm E :: C : D \pm F$.

Cor. If $A : B :: C : D$, and $E : B :: F : D$; then $A \pm E : B :: C \pm F : D$. For, by alternating the analogies, $A : C :: B : D$, and $E : F :: B : D$; whence $B : D :: A \pm E : C \pm F$, and, by alternation and inversion, $A \pm E : B :: C \pm F : D$.

PROP. XXI. THEOR.

In continued proportionals, the difference between the first and second is to the first, as the difference between the first and last terms to the sum of all the terms, excepting the last.

Let $A : B :: B : C :: C : D :: D : E$; then if $A > B$, $A - B : A : A - E : A + B + C + D$.

For (V. 19.), $A : B :: A+B+C+D : B+C+D+E$, and consequently (V. 11. cor.), $A-B : A :: (A+B+C+D)-(B+C+D+E) : A+B+C+D$; that is, omitting $B+C+D$ in the third term, $A-B : A :: A-E : A+B+C+D$.

If $A < B$, then $B-A : A :: (B+C+D+E)-(A+B+C+D) : A+B+C+D$, that is, $B-A : A :: E-A : A+B+C+D$.

The same reasoning, it is evident, will hold for any number of terms.

PROP. XXII. THEOR.

The products of the like terms of any numerical proportions, are themselves proportional.

$$\text{Let } A : B :: C : D$$
$$E : F :: G : H$$
$$I : K :: L : M;$$

then $AEI : BFK :: CGL : DHM$.

For (V. 6.), from the first analogy $AD = BC$, from the second analogy $EH = FG$, and from the third analogy $IM = KL$; whence the compound product $AD.EH.IM = BC.FG.KL$. But $AD.EH.IM = AEI.DHM$ (V. 2.), and $BC.FG.KL = BFK.CGL$; wherefore $AEI.DHM = BFK.CGL$, and consequently (V. 6.) $AEI : BFK :: CGL : DHM$.

The same reasoning, it is obvious, applies to any number of proportionals.

Cor. 1. Hence the powers of the successive terms of numerical proportions, are likewise proportional. For, if $A : B :: C : D$, and, repeating the analogy, $A : B :: C : D$; then, by multiplication, $AA : BB :: CC : DD$, or $A^2 : B^2 :: C^2 : D^2$.

Again, let $A : B :: C : D$, and, repeating the analogy,

$$A : B :: C : D,$$

and $A : B :: C : D$; whence, by multiplying the corresponding terms,

$$A^3 : B^3 :: C^3 : D^3.$$

And so the induction may be pursued generally,

$A^m : B^m :: C^m : D^m$

Cor. 2. Hence also the roots of the terms of a numerical proportion, are proportional. If $A:B::C:D$, then $\sqrt{A}:\sqrt{B}::\sqrt{C}:\sqrt{D}$. For let $\sqrt{A}:\sqrt{B}::\sqrt{C}:\sqrt{E}$, and, by the last corollary, $A:B::C:E$; but $A:B::C:D$, whence $C:E::C:D$, and consequently $E=D$, or $\sqrt{A}:\sqrt{B}::\sqrt{C}:\sqrt{D}$.—In the same manner, it may be shown in general that, if $A:B::C:D$, $\sqrt[n]{A}:\sqrt[n]{B}::\sqrt[n]{C}:\sqrt[n]{D}$.

PROP. XXIII. THEOR.

The ratio which is conceived to be compounded of other ratios, is the same as that of the products of their corresponding numerical expressions.

Suppose the ratio of $A:D$ is compounded of $A:B$, of $B:C$, and of $C:D$, and let $A:B::K:L$, $B:C::M:N$, and $C:D::O:P$; then will $A:D::KMO:LNP$.

For, since $A:B::K:L$,
$B:C::M:N$,
and $C:D::O:P$,

the products of the similar terms are proportional (V. 22.), or $ABC:BCD::KMO:LNP$. But $A:D::ABC:BCD$ (V. 3.), and consequently $A:D::KMO:LNP$.

The same mode of reasoning is applicable to any number of component ratios.

PROP. XXIV. THEOR.

A duplicate ratio is the same as the ratio of the second powers of the terms of its numerical expression, and a triplicate ratio is the same as the third powers of those terms.

The duplicate ratio of $A:B$ is denoted by $A^2:B^2$, and the triplicate ratio by $A^3:B^3$.

For the duplicate ratio of $A:B$, being the double com-

pound of A : B and of A : B, is (V. 22.) the same as that of the corresponding products A.A : B.B, or $A^2 : B^2$.

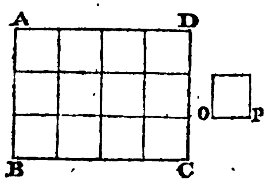
Again, the triplicate ratio of A : B, being the triple compound of A : B, of A : B, of A : B, is the same as that of the corresponding products AAA : BBB, or $A^3 : B^3$.

Cor. Hence the subduplicate ratio of A : B, is $\sqrt{A} : \sqrt{B}$, and the subtriplicate ratio of A : B, is $\sqrt[3]{A} : \sqrt[3]{B}$.

PROP. XXV. THEOR.

The product of the numbers expressing the sides of a rectangle, will represent its quantity of surface, as measured by a square described on the linear unit.

Let ABCD be a rectangle and OP the linear measure; and suppose the side AB to contain OP, *m* times, and the side BC to contain it, *n* times. Divide these sides accordingly (I. 38.), and, through the points of section, draw straight lines (I. 24.) parallel to AD and DC : the whole rectangle will thus be divided into cells, each of them equal to the square of OP.



It is evident, that there stand on BC, *n* columns, and that each of these columns contains, *m* cells; consequently the entire space includes, *m.n* cells, or is equal to the square of OP repeated *mn* times.

Cor. 1. If $m=n$, then AB=BC, and the rectangle becomes a square; but *mn* is in that case equal to *nn*, or n^2. Whence the surface of a square is equal to the second power of the number denoting its side.

Cor. 2. Rectangles which have the same altitude *m* are as their bases *n* and *p*; for (V. 3. $mn : mp :: n : p$. And triangles having the same altitude, being (I. 27. cor.) the halves of these rectangles, must likewise be as their bases.

Cor. 3. If two rectangles be equal, their respective sides are reciprocally proportional, or form the extremes and means of an analogy. For if $mn=pq$, then (V. 6.) $m:p::q:n$.

PROP. XXVI. THEOR.

If three straight lines be in continued proportion, the first is to the third, as the square of the sum or difference of the first and second to the square of the sum or difference of the second and third.

Let $A:B::B:C$; then $A:C::(A+B)^2:(B+C)^2$, and $A:C::(A-B)^2:(B-C)^2$, or $(B-A)^2:(C-B)^2$.

For (V. 19. cor. 2.) $A:B::A\pm B:B\pm C$, and consequently (V. 22. cor. 1. and V. 25. cor. 1.) $A^2:B^2::(A\pm B)^2:(B\pm C)^2$. But (V. 24.) $A:C::A^2:B^2$; wherefore $A:C::(A\pm B)^2:(B\pm C)^2$.

Cor. The converse of this proposition is likewise true.

PROP. XXVII. PROB.

Given two homogeneous quantities, to find, if possible, their greatest common measure.

Let it be required to find the greatest common measure, that two quantities A and B, of the same kind, will admit.

Supposing A to be greater than B, take B out of A, till the remainder C be less than it; again, take C out of B, till there remain only D; and continue this alternate operation, till the last divisor, suppose E, leave no remainder whatever; E is the greatest common measure of the quantities proposed.

For, that which measures B will measure its multiple; and being a common measure, it also measures A, and measures, therefore, the difference between the multiple of B and A (V. 1. cor. 1.), that is, C; the required measure, hence, measures the multiple of C, and consequently the difference of this multiple and B, which it measured,—that is D: And lastly,

this measure, as it measures the multiple of D, must consequently measure the difference of this from C, or it must measure E. Here the decomposition is presumed to terminate. Wherefore, the common measure of A and B, since it measures E, may be E itself; and it is also the greatest possible measure, for nothing greater than E can be contained in this quantity.

By retracing the steps likewise, it might be shown, that E measures, in succession, all the preceding terms D, C, B, and A.

If the process of decomposition should never come to a close, the quantities A and B do not admit a common measure,—or they are *incommensurable*. But, as the residue of the subdivision is necessarily diminished at each step of this operation, it is evident that an element may be always discovered, which will measure A and B nearer than any assignable difference whatever.

PROP. XXVIII. PROB.

To express by numbers, either exactly or approximately, the ratio of two given homogeneous quantities.

Let A and B be two quantities of the same kind, whose numerical ratio it is required to discover.

Find, by the last Proposition, the greatest common measure E of the two quantities; and let A contain this measure K times, and B contain it L times: Then will the ratio K : L express the ratio of A : B.

For the numbers K and L severally consist of as many units, as the quantities A and B contain their measure E. It is also manifest, since E is the greatest possible divisor, that K and L are the smallest numbers capable of expressing the ratio of A to B.

If A and B be incommensurable quantities, their decomposition is capable at least of being pushed to an unlimited extent; and, consequently, a divisor can always be found so extremely minute, as to measure them both to any degree of precision.

Otherwise thus.

But the numerical expression of the ratio A : B, may be deduced indirectly, from the series of quotients obtained in the operation for discovering their common measure.

Let A contain B, m times, with a remainder C; B contain C, n times, with a remainder D; and, lastly, suppose C to contain D, p times, with a remainder E, and which is contained in D, q times exactly. Then $D=qE$, $C=pD+E$, $B=nC+D$, and $A=mB+C$; whence the terms D, C, B, and A, are successively computed, as multiples of E;—A and B will, therefore, be found to contain E their common measure K and L times, or the numerical expression for the ratio of those quantities, is K : L *.

PROP. XXIX. THEOR.

A straight line is incommensurable with its segments formed by medial section.

If the straight line AB be cut in C, such that the rectangle AB, BC is equivalent to the square of AC; no part of AB, however small, will measure the segments AC, BC.

For (V. 27.) take AC out of AB, and again the remainder BC out of AC. But AD, be-

ing made equal to BC, the straight line AC is likewise divided in D, by a medial section (II. 22. cor. 1.); and, for the same reason, taking away the successive remainders CD, or AE, from AD, and DE or AF from AE, the subordinate lines AD and

* See Note XXXVI.

AE are also divided medially in the points E and F. This operation produces, therefore, a series of decreasing lines, all of them divided by medial section: Nor can the process of decomposition ever terminate; for though the remainders BC, CD, DE, and EF thus continually diminish, they still must constitute the segments of a similar division. Consequently there exists no final quantity which would measure both AB and AC.

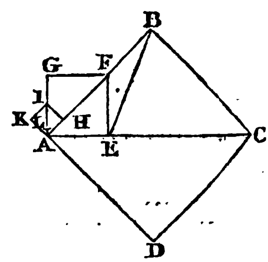
PROP. XXX. THEOR.

The side of a square is incommensurable with its diagonal.

Let ABCD be a square and AC its diagonal; AC and AB are incommensurable.

For make CE equal to AB or BC, draw (I. 5. cor.) the perpendicular EF, and join BE.

Because CE is equal to BC, the angle CEB (I. 11.) is equal to CBE; and since CEF and CBF are right angles, the remaining angle BEF is equal to EBF, and the side EF (I. 12.) equal to BF; but EF is also equal to AE, for the angles EAF and EFA of the triangle AEF are evidently each half a right angle. Whence, making FH equal to FB, FE or AE,—the excess AE of the diagonal AC above the side AB, is contained twice in AB, with a remainder AH; and AH again, being the excess of the diagonal AF of the square GE above the side AE, must, for the same reason, be contained twice in AG, with a new remainder AL; and this remainder will likewise be contained twice in AH, the side of the square KH. This process of subdivision is, therefore, interminable, and the same relations are continually reproduced *.



* See Note XXXVII.

ELEMENTS

OF

GEOMETRY.

BOOK VI.

THE doctrine of Proportion, grounded on the simplest theory of numbers, furnishes a most powerful instrument, for abridging and extending mathematical investigations. It easily unfolds the primary relations of figures, and the sections of lines and circles; but it also discloses with admirable felicity that vast concatenation of general properties, not less important than remote, which, without such aid, might for ever have escaped the penetration of the geometer. The application of Arithmetic to Geometry forms, therefore, one of those grand epochs which occur, in the lapse of ages, to mark and accelerate the progress of scientific discovery.

DEFINITIONS.

1. Straight lines which proceed from the same point, are termed *diverging* lines.

2. Straight lines are divided *similarly*, when their corresponding segments have the same ratio.

3. A straight line is said to be cut *harmonically*, if it consist of three segments, such that the whole line is to one extreme, as the other extreme to the middle part.

4. The *area* of a figure is its surface, or the quantity of space which it occupies.

5. *Similar* figures are such as have their angles respectively equal, and the containing sides proportional.

6. If two sides of a rectilineal figure be the extremes of an analogy, of which the means are two sides containing an equal angle in another rectilineal figure; these sides are said to be *reciprocally* proportional.

PROP. I. THEOR.

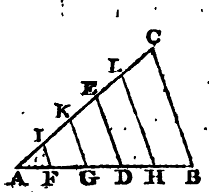
Parallels cut diverging lines proportionally.

The parallels DE and BC cut the diverging lines AB and AC into proportional segments.

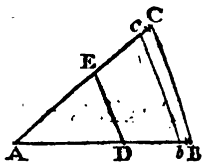
Those parallels may lie on the same side of, the vertex, or on opposite sides; and they may consist of two, or of more lines.

1. Let the two parallels DE and BC intersect the diverging lines AB and AC, on the same side of the vertex A; then are AB and AC cut proportionally, in the points D and E,—or AD : AB : : AE : AC.

For if AD be commensurable with AB, find (V, 27.) their common measure M, and, from the corresponding points of section in AD and AB, draw (I. 24.) the parallels FI, GK, and HL. It is evident, from Book I. Prop. 38. that these parallels will also divide the straight lines AE and AC equally. Wherefore the measure M, or AF the submultiple of AD, is contained in AB, as often as AI, the like submultiple of AE, is contained in AC; consequently (V. def. 10.) the ratio of AD to AB is the same with that of AE to AC.



But, should the segments AD and AB be incommensurable, they may still be expressed numerically, and this to any required degree of precision. AD being divided (I. 38.) into equal parts, these parts, continued towards B, will, together with a residuary portion, compose the whole of AB. Let this division of AD extend in DB to *b*, and draw the parallel *bc*. If the parts of AD and AB be again subdivided, the corresponding residue will evidently be diminished; and thus, at

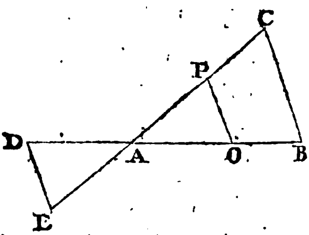


each successive subdivision, the terminating parallel *bc* must approximate perpetually to BC. Wherefore, by continuing this process of exhaustion, the divided lines *Ab* and *Ac* will approach the limits AB and AC, nearer than any finite or assignable interval. Consequently, from the preceding demonstration, AD : AB : : AE : AC.

And since AD : AB : : AE : AC, it follows, by conversion (V. 11.), that AD : DB : : AE : EC, and again, by composition (V. 9.), that AB : DB : : AC : EC.

2. Let the two parallels DE and BC cut the diverging lines DB and EC, on opposite sides of A; the segments AB, AD have the same ratio with AC, AE,—or AB : AD : : AC : AE.

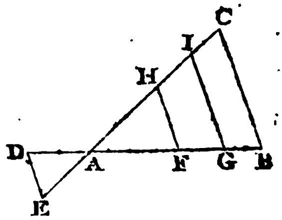
For, make AO equal to AD, AP to AE, and join OP. The triangles APO and AED, having the sides AO, AP equal to AD, AE, and the contained vertical angle OAP equal to DAE, are equal (I. 3.), and consequently the angle AOP is equal to ADE; but these being alternate angles, the straight line OP (I. 23.) is parallel to DE or BC, and hence, from what was already demonstrated, AB : AO or AD : : AC : AP or AE.



And since AB : AD : : AC : AE, by composition BD : AD : : CE : AE, and, by conversion, BD : AB : : CE : AC.

3. Lastly, let more than two parallels, BC, DE, FH, and GI, intersect the diverging lines AB and AC; the segments DA, AF, FG, and GB, in DB, are proportional respectively to EA, AH, HI, and IC, the corresponding segments in EC.

For, from the second case, AD : AF : : AE : AH; and, from the first case, AF : FG : : AH : HI. But from the same case, AG : FG : : AI : HI, and AG : GB : : AI ; IC; whence (V. 15.) FG : GB : : HI : IC.



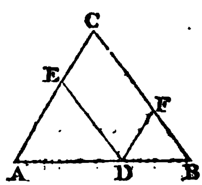
Cor. 1. Hence the converse of the proposition is also true, or that straight lines which cut diverging lines proportionally are parallel; for it would otherwise follow, that a new division of the same line would not alter the relation among the segments, which is evidently absurd.

Cor. 2. Hence, if the segments of one diverging line be equal to those of another, the straight lines which join them are parallel.

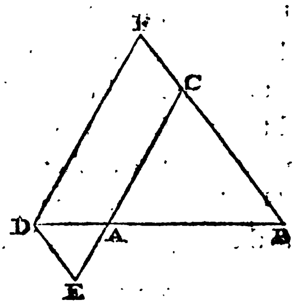
PROP. II. THEOR.

Diverging lines are proportional to the corresponding segments into which they divide parallels.

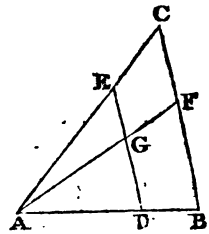
Let two diverging lines AB and AC cut the parallels BC and DE; then AB : AD : : BC : DE.



For draw DF parallel to AC. And, by the last Proposition, the parallels AC and DF must cut the straight lines AB and BC proportionally, or AB : AD : : BC : CF. But CF is equal (I. 27.) to the opposite side DE of the parallelogram DECF; and consequently AB : AD : : BC : DE.



Next, let more than two diverging lines AB, AF, and AC intersect the parallels BC and DE; the segments BF and FC have respectively to DG and GE the same ratio as AB has to AD.



From what has been already demonstrated, it appears, that AB : AD : : BF : DG, and also that AF : AG : : FC : GE. But by the last Proposition, AB : AD : : AF : AG; wherefore AB : AD : : FC : GE. The same mode of reasoning, it is obvious, might be extended to any number of sections. Whence AB : AD : : BF : DG : : FC : GE.

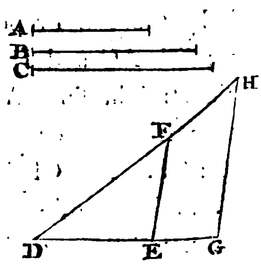
Cor. 1. Hence the straight lines which cut diverging lines equally, being parallel (VI. 1. cor. 2.), are themselves proportional to the segments intercepted from the vertex.

Cor. 2. Hence parallels are cut proportionally by diverging lines *.

PROP. III. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, and C be three straight lines, to which it is required to find a fourth proportional.



Draw the diverging lines DG and DH, make DE equal to A, DF to B, and DG to C, join EF, and through G draw (I. 24.) GH parallel to EF and meeting DH in H; DH is a fourth proportional to the straight lines A, B, and C.

* See Note XXXVIII.

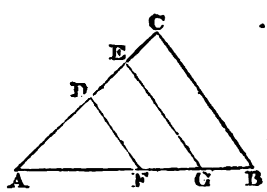
For the diverging lines DG and DH are cut proportionally by the parallels EF and GH (VI. 1.), or DE : DF : : DG : DH, that is, A : B : : C : DH.

Cor. If the mean terms B and C be equal, it is obvious that DG will become equal to DF, and that DH will be found a third proportional to the two given terms A and B.

PROP. IV. PROB.

To cut a given straight line into segments, which shall be proportional to those of a divided straight line.

Let AB be a straight line, which it is required to cut into segments proportional to those of a given divided straight line.



Draw the diverging line AC, and make AD, DE, and EC, equal respectively to the segments of the divided line, join CB, and draw EG and DF parallel to it (I. 24.) and meeting AB in F and G; AB is cut in those points proportionally to the segments of AC.

For the parallels DF, EG, and CB must cut the diverging lines AB and AC proportionally (VI. 1.), or AF : FG : : AD : DE, and FG : GB : : DE : EC.

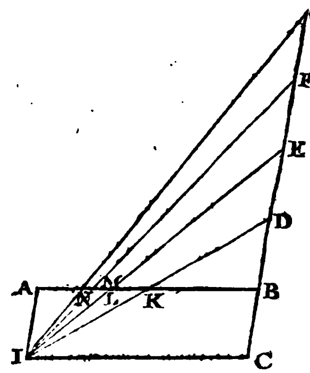
PROP. V. PROB.

To cut off the successive parts of a given straight line.

Let AB be a straight line from which it is required to cut off successively the half, the third, the fourth, the fifth, &c.

Through B draw the inclined straight line CBG extended both ways, in this take any point C, and make BD, DE, EF,

FG, &c. each equal to BC, complete the parallelogram ABCI, and join ID, IE, IF, IG, &c. cutting AB in the points K, L, M, N, &c.; then is the segment AK the half of AB, AL the third, AM the fourth, and AN the fifth part, of the same given line.

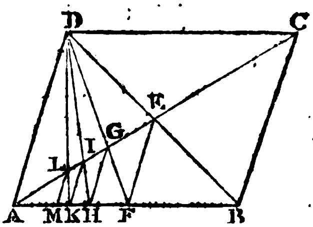


For the segments of the straight line AB must be proportional to the segments of the parallels AI and BG, intercepted by the diverging lines ID, IE, IF, IG, &c. Thus, AK : KB : : AI : BD; but, by construction, BC or AI=BD, whence (V. 4.) AK=KB, and therefore AK is the half of AB. Again, AL : LB : : AI : BE; and since BE=2AI, it follows, that LB=2AL, or AL is the third part of AB. In the same manner, AM : MB : : AI : BF; but BF=3AI, whence MB=3AM, or AM is the fourth part of AB. And, by a like process, it may be shown that AN is the fifth part of AB.

Otherwise thus.

On AB describe the rhomboid ABCD, and through E, the intersection of its diagonals AC and BD, draw EF parallel to AD (I. 24.), join DF, and through G, where it cuts AC, draw GH likewise parallel to AD, again join DH and draw the parallel IK, and so repeat the operation: Then will AF be the half of AB, AH the third, AK the fourth, and AM the fifth part of it.

Because AD and EF are parallel, DE : EB : : AF : FB (VI. 1.); but DE=EB (I. 29.), wherefore AF=FB, or AF is the half of AB. And AD and EF being intercepted parallels, AD : EF : : AB : BF (VI. 2.); consequently since



AB is double of BF, AD is likewise double of EF (V. 5.).—Again, the diverging lines AGE and DGF are proportional to the intercepted parallels AD and EF (VI. 2.), or AD : EF : : AG : GE; and GH being parallel to EF, AG : GE : : AH : HF (VI. 1.), whence AD : EF : : AH : HF; but AD was shown to be double of EF, wherefore AH is double of HF (V. 5.), or AH is two-thirds of AF, or of the half of AB, and is consequently the third part of the whole AB. And, since AF : HF : : AD : GH and AF is triple of HF, it is evident that AD is triple of GH; but AD : GH : : AI : IG : : AK : KH, and, AD being triple of GH, AK must also be triple of KH; or AK is three-fourths of AH, which was proved to be the third of AB, whence the segment AK is the fourth part of the whole line AB. By a like process, it is shown that AM is the fifth part of AB *.

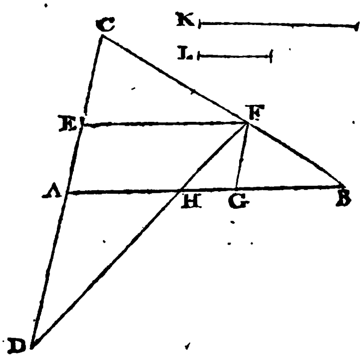
PROP. VI. PROB.

To divide a straight line harmonically, in a given ratio.

Let AB be a straight line, which it is required to cut harmonically, in the ratio of K to L.

Through A draw the diverging line AC, and produce it both ways till AC and AD be each equal to K, make AE

* See Note XXXIX.



equal to L, join CB, draw EF parallel to AB, and FG parallel to CA, and join DF; the straight line AB is divided harmonically in the points H and G, such that K : L : : AB : BG : : AH : HG.

For the parallels AC and GF, being intercepted by the diverging lines AB and CB, AC : GF : : AB : BG (VI. 2.). Again, the diverging lines AG and DF are cut by the parallels AD and FG, whence (VI. 1.) AD : GF : : AH : HG. Wherefore, AB : BG : : AH : HG; and each of these ratios is the same as that of AC or AD to GF, or that of K to L.

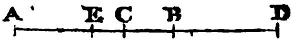
Cor. Hence AG is divided, internally in H and externally in B, in the same ratio. In like manner, BH is divided proportionally, by an external and internal section in A and G; for AB : BG : : AH : HG, and alternately AB : AH : : BG : HG.

PROP. VII. THEOR.

If a straight line be divided internally and externally in the same ratio, half the line is a mean proportional between the distances of the middle from the two points of unequal section.

Let the straight line AB be divided in the same ratio, internally and externally in C and D, and also be bisected in

E; the half EB is a mean proportional between EC and ED, or EC : EB : : EB : ED.



For since AC : CB : : AD : DB, by mixing and inversion AC—CB : AC+CB : : AD—DB : AD+DB, that is, 2EC : AB : : AB : 2ED, and, halving all the terms of the analogy, (V. 3.) EC : EB : : EB : ED.

Cor. Hence if a straight line be cut internally and externally in the same ratio, the square of the interval between the points of section is equivalent to the difference between the rectangles under the internal and external segments. For (II. 19. cors.) $AD.DB = ED^2 - EB^2$, and $AC.CB = EB^2 - EC^2$; consequently $AD.DB - AC.CB = ED^2 - 2EB^2 + EC^2$, or (V. 6.) $ED^2 - 2ED.EC + EC^2$, which (II. 18.) is the square of ED—EC or of CD.—By a similar procedure, the converse of the proposition and its corollary may be established.

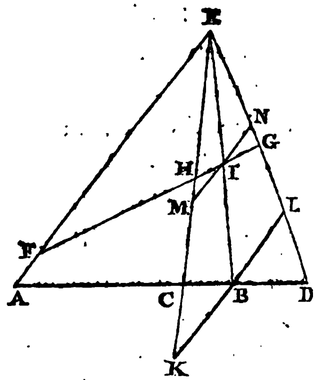
PROP. VIII. THEOR.

If diverging lines divide a straight line harmonically, they will cut every intercepted straight line also in harmonic proportion.

Let the diverging lines EA, EC, EB, and ED terminate in the harmonic section of the straight line AD; any intercepted straight line FG will be likewise cut by them harmonically, or FG : GI : : FH : HI.

For, through the points B and I, draw (I. 24.) KL and MN parallel to AE.

Because the parallels AE and BL are intercepted by the diverging lines DA and DE, AD : DB : : AE : BL (VI. 2.); and for the same reason, the parallels AE and BK being in-



tercepted by the diverging lines AB and EK, AC : CB : : AE : BK. And since AD is divided harmonically, AD : DB : : AC : CB; wherefore AE : BL : : AE : BK, and consequently BL = BK. But, KL being parallel to MN, BL : BK : : IN : IM (VI. 2. cor. 2.); consequently, BL being equal to BK, IN must also be equal to IM (V. 4.); whence FE : IN : : FE : IM. Again, FE : IN : : FG : GI, for the parallels FE and IN are cut by the diverging lines GF and GE; and FE : IM : : FH : HI, since the parallels FE and IM are cut by the diverging lines FI and EM. Wherefore, by identity of ratios, FG : GI : : FH : HI; or the intercepted straight line FG is cut harmonically in the points H and I.

PROP. IX. THEOR.

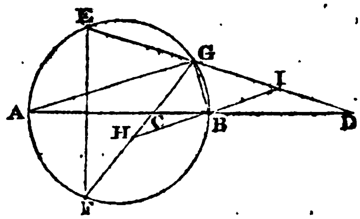
If from any point in the circumference of a circle, straight lines be drawn to the extremities of a chord and meeting the perpendicular diameter, they will divide that diameter, internally and externally, in the same ratio.

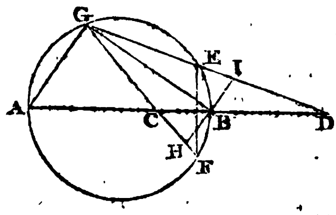
Let the chord EF be perpendicular to the diameter AB of a circle, and from its extremities F and E straight lines FG and EG be inflected to a point G in the circumference, and cutting the diameter internally and externally in C and D; then will AC : CB : : AD : DB.

For join AG and BG, and draw HBI parallel to AG.

Because AEGB is a semicircle, the angle AGB is a right

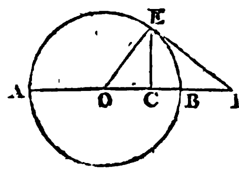
angle (III. 22.); wherefore AG and HI being parallel, the alternate angle GBI is right (I. 23.), and likewise its adjacent angle GBH. But the diameter AB, being perpendicular to the chord EF, must (III. 4. and 15.) bisect the arc FAE, and therefore the angle EGA is equal to AGF (III. 13. cor.) or (III. 19.), its supplement. And since AG is parallel to HI, the angle EGA is equal to the angle GIB or its supplement (I. 23.); and, for the same reason, the angle AGF is equal to the alternate angle GHB. Whence the angle GIB is equal to GHB; but the angles GBI and GBH being both right angles, are equal, and the side GB is common to the two triangles BIG and BHG, which are, therefore, equal (I. 21.), and consequently BH is equal to BI, and AG : BH : : AG : BI. Now, because the parallels AG and BH are intercepted by the diverging lines AB and GH, AG : BH : : AC : CB (VI. 2.); and since the parallels AG and BI are intercepted by the diverging lines GD and AD, AG : BI : : AD : DB. Wherefore, by identity of ratios, AC : CB : : AD : DB, that is, the straight line AB is cut in the same ratio, internally and externally, or the whole line AD is divided harmonically in the points C and B.





Cor. 1. As the points E and G come nearer each other, it is obvious that the straight line EGD will approach continually to the position of the tangent, which is its ultimate limit. Hence the tangent and the perpendicular, from the point of contact or mutual coincidence, cut the diameter pre-

portionally, or AC : CB : : AD : DB. It is, therefore, evident (VI. 7.) that, O being the centre, OC : OB : : OB : OD.



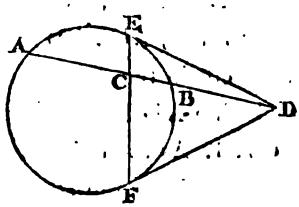
Cor. 2. Since OC : OB : : OB : OD, it follows (V. 19. cor. 2.) that OC : OD : : OB^2—OC^2 or AC.CB : OD^2—OB^2 or AD.DB; whence, by division, CD : OD : : AD.DB—AC.CB, or (VI. 7. cor.) CD^2 : AD.DB *.

PROP. X. THEOR.

A straight line drawn from the concourse of two tangents to the concave circumference of a circle, is divided harmonically, by the convex circumference and the chord which joins the points of contact.

Let ED and FD be two tangents applied to the circle AEBF; the secant DA, drawn from their point of concourse, will be cut in harmonic proportion, by the convex circumference EBF and the chord EF which joins the points of contact,—or AD : DB : : AC : CB.

For the tangents ED and FD are equal (III. 26. cor.), and EDF being thus an isosceles triangle, $DE^2 = DC^2 +$ EC.CF (II. 23.); (but III. 32.) DE^2 is also equal to AD.DB, and the chords AB and EF, by their mutual intersection, make the rectangle EC, CF equal to AC, CB. Whence DC^2=AD.DB—AC.CB, and therefore (VI. 7. cor.) AC : CB : : AD : DB.



* See Note XI.

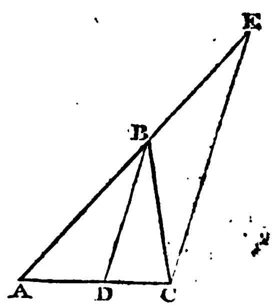
PROP. XI. THEOR.

A straight line which bisects, either internally or externally, the vertical angle of a triangle, will divide its base into segments, internal or external, that are proportional to the adjacent sides of the triangle.

Let the straight line BD bisect the vertical angle of the triangle ABC; it will cut the base AC into segments which have the same ratio as the adjacent sides, or AD : DC : : AB : BC.

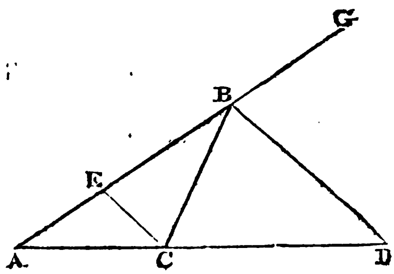
For through C draw CE parallel to DB (I. 24.), and meeting the production of AB in E.

Because DB and CE are parallel, the exterior angle ABD is equal to BEC, and the alternate angle DBC equal to BCE (I. 23.); wherefore the angle ABD being equal by hypothesis to DBC, the angle BEC is equal to BCE, and consequently (I. 12.) the triangle CBE is isosceles, or BE is equal to BC. But the parallels DB and CE cut the diverging lines AC and AE proportionally (VI. 1.), or AD : DC : : AB : BE; that is, since BE=BC, AD : DC : : AB : BC.



Again, let the vertical line BD bisect the exterior angle CBG of the triangle; it will divide the base into external segments AD and DC, which are also proportional to the adjacent sides AB and BC.

For through C draw CE parallel to DB, and meeting AB in E.



The equal angles GBD and DBC are, from the properties of parallel straight lines, respectively equal to BEC and BCE, and consequently the triangle CBE is isosceles, or the side BC is equal to BE. And since the diverging lines AD and AB are cut by the parallels DB and CE proportionally, AD : DC : : AB : BE or BC.

Cor. Hence the converse of the Proposition is likewise true, or if a straight line be drawn from the vertex of a triangle to cut the base in the ratio of the adjacent sides, it will bisect the vertical angle; for it is evident, from VI. 6. cor., that a straight line is only capable of a single section, whether internal or external, in a given proportion.

Scholium. The vertical line BD must bisect the base AC of the triangle, when the sides AB and BC are equal. In the case where BD bisects the exterior angle CBG, if AB be supposed to approach to an equality with BC, the straight line EC will come nearer to AC, and consequently the incidence D of the parallel BD with AC will be thrown continually more remote. But when the side AB is equal to BC, the straight line BD, being now parallel to AC, will never meet it, or there can be no equality of external section; for though the ratio of AD to CD tends towards the ratio of equality as the point D retires, yet the constant difference AC between those distances must always bear a sensible relation to them. After BD, in turning about the point B, has passed the limits of distance beyond C, it re-appears in an opposite direction beyond A, when AB, receding from equality, has become less than BC *.

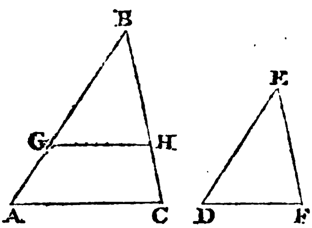
* *See Note XLI.*

PROP. XII. THEOR.

Triangles are similar, which have their corresponding angles equal.

Let the triangles ABC and DEF, have the angle CAB equal to FDE, CBA to FED, and consequently (I. 32.) the remaining angle BCA equal to EFD; these triangles are similar, or the sides in both which contain equal angles are proportional.

For make BG equal to ED, and draw GH parallel to AC.



Because GH is parallel to AC, the exterior angle BGH is equal (I. 23.) to BAC, that is, to EDF; and the angle at B is, by hypothesis, equal to that at E, and the interjacent side BG was made equal to ED; wherefore (I. 21.) the triangle GBH is equal to DEF. But, the diverging lines BA and BC being cut proportionally by the parellels AC and GH (VI. 1.), AB is to BC as BG to BH, or as ED to EF. Again, those diverging lines being proportional to the intercepted segments AC and GH of the parallels (VI. 2.), AB is to BG as AC is to GH, and alternately AB is to AC as BG is to GH, or as ED to DF. In the same manner, as BC is to BH so is AC to GH, and alternately, as BC is to AC so is BH or EF to GH or DF. And thus, the sides opposite to equal angles in the triangles ABC and DEF, are the homologous terms of a proportion.

Cor. Isosceles triangles are similar which have their vertical angles equal. For (I. 32.) the supplementary angles at the base must be together equal, and consequently they are equal to each other.

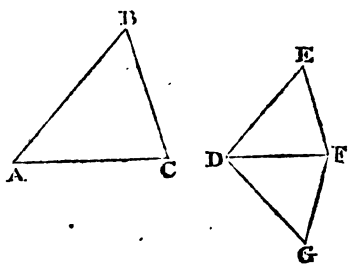
PROB. XIII. THEOR.

Triangles which have the sides about two of their angles proportional, are similar.

In the triangles ABC and DEF, let AB : AC : : DE : DF and BC : AC : : EF : DF; then is the angle BAC equal to EDF, and the angle BCA to EFD.

For (I. 4.) draw DG and FG, making angles FDG and DFG equal to CAB and ACB.

By the last Proposition, the triangle ABC is similar to DGF, and consequently AB : AC : : DG : DF; but, by hypothesis, AB : AC : : DE : DF, and hence, from identity of ratios, DG : DF : : DE : DF, or DG is equal to DE. In the same manner, BC : AC : : EF : DF, and BC : AC : : GF : DF; whence EF : DF : : GF : DF, and EF is equal to FG. Wherefore the triangles DEF and DGF, having thus the sides DE and EF equal to DG and FG, and the side DF common to both, are (I. 2.) equal; consequently the angle EDF is equal to FDG or BAC, and the angle EFD is equal to DFG or BCA.



Cor. Hence isosceles triangles which have either side proportional to the base, are similar.

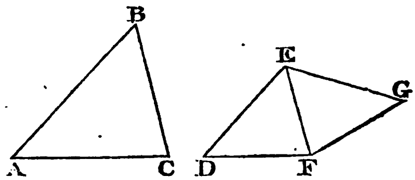
PROP. XIV. THEOR.

Triangles are similar, if each have an equal angle and its containing sides proportional.

In the triangles BAC and EDF, let the angle ABC be equal to DEF, and the sides which contain the one be proportional to those which contain the other, or AB : BC : : DE : EF; the triangles BAC and EDF are similar.

For, from the points E and F, draw EG and FG, making the angles FEG and EFG equal to CBA and BCA.

The triangles BAC and EGF, having thus their corresponding angles equal, are similar (VI. 12.), and therefore AB : BC : : EG : EF. But by hypothesis, AB : BC : : ED : EF; wherefore EG : EF : : ED : EF, and consequently EG is equal to ED. Hence the triangles GFE and DFE, having the side EG equal to ED, EF common to both, and the contained angle GEF equal to ABC or DEF, are equal (I. 3.), and therefore the angle EFG or BCA is equal to EFD; consequently the remaining angles BAC and EDF of the triangles ABC and DEF, are equal (I. 32.), and these triangles are (VI. 12.) similar.

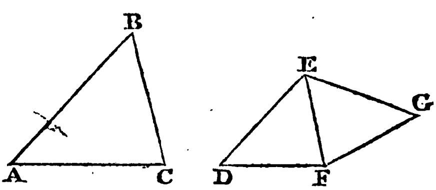


PROP. XV. THEOR.

Triangles are similar, which, being of the same affection, have each an equal angle, and the sides containing another angle proportional.

Let the triangles ABC and DEF, which are of the same affection, have the angle ABC equal to DEF and the sides that contain the angles at C and F proportional, or BC : AC : : EF : FD; the triangles ABC and DEF are similar.

For, from the points E and F draw EG and FG, making the angles FEG and EFG equal to ABC and BCA.



The triangle ABC is evidently similar to GEF, and BC :

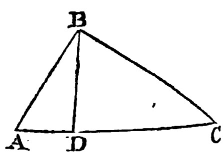
CA : : EF : FG; but, by hypothesis, BC : CA : : EF : FD, and therefore EF : FG : : EF : FD, and FG is equal to FD. Whence the triangles EGF and EDF, having the side FG equal to FD and the side EF common, and being both of the same affection with CAB, are equal (I. 22.); consequently the angle GFE is equal to DFE or ACB, and therefore (VI. 12.) the triangles ABC and DEF are similar.

PROP. XVI. THEOR.

A perpendicular let fall upon the hypotenuse of a right-angled triangle from the opposite vertex will divide it into two triangles which are similar to the whole and to each other.

Let the triangle ABC be right-angled at B, from which the perpendicular BD falls upon the hypotenuse AC; the triangles ABD and DBC, thus formed, are similar to each other, and to the whole triangle ACB.

For the triangles ABD and ACB, having the angle BAC common, and the right angle ADB equal to ABC, are similar (VI. 12.). Again, the triangles DBC and ACB are similar, since they have the angle BCD common, and the right angle BDC equal to ABC. The triangles ABD and DBC being, therefore, both similar to the same triangle ABC, are evidently similar to each other (VI. 12.).



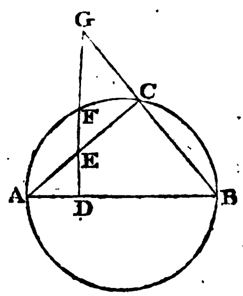
Cor. 1. Hence the side of a right-angled triangle is a mean proportional between the hypotenuse and the adjacent segment, formed by a perpendicular let fall upon it from the opposite vertex; and the perpendicular itself is a mean proportional between those segments of the hypotenuse. For the triangles ABC and ADB being similar, AC : AB : : AB : AD; and the triangles ABC and BDC being similar, AC : BC : : BC : CD; again, the triangles ADB and BDC are similar, and therefore AD : DB : : DB : DC.

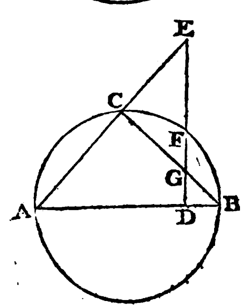
Cor. 2. If the hypotenuse and the sides of a right-angled triangle form a continued proportion, the hypotenuse will be divided in extreme and mean ratio, by the perpendicular let fall upon it from the opposite vertex. For, by the last Corollary, AC : AB : : AB : AD, and therefore (V. 6.) $AB^2 = AC.AD$; in like manner, AC : BC : : BC : CD. But, by hypothesis, AC : BC : : BC : AB; whence BC : CD : : BC : AB, and consequently AB=DC, and $AB^2 = AC.AD = CD^2$. Wherefore (V. 6.) AC : CD : : CD : AD *.

PROP. XVII. THEOR.

The perpendicular within a circle, is a mean proportional to the segments formed on it by straight lines, drawn from the extremities of the diameter, through any point in the circumference.

Let the straight lines AEC and BCG, drawn from the extremities of the diameter of a circle through a point C in the circumference, cut the perpendicular to AB; the part DF within the circle is a mean proportional between the segments DE and DG.





For the angle ACB, being in a semicircle, is a right angle (III. 22.), and the angle ABG is common to the two triangles ABC and GBD, which are, therefore, similar (VI. 12.). Hence the remaining angle BAC is equal to BGD, and consequently the triangles ADE and GDB are similar; wherefore AD : DE : : DG : DB, and (V. 6.) AD.DB=DE.DG. But (III. 32. cor.), the rectangle under AD and DB is equivalent to the

* See Note XLII.

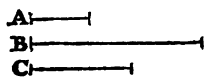
square of DF; whence DE.DG$=$DF^2, and (V. 6.) DE : DF :: DF : DG *.

PROP. XVIII. PROB.

To find the mean proportional between two given straight lines.

Let it be required to find the mean proportional between the straight lines A and B.

Find C (III. 33.) the side of a square which is equivalent to the rectangle contained by A and B; C is the mean proportional required.

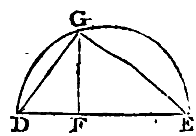


For since C^2=AB, it follows (V. 6.) that A : C : : C : B.

Otherwise thus.

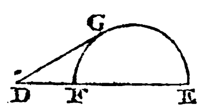
Make DF$=$A and DE$=$B, on DE describe the semicircle DGE, draw FG perpendicular to the diameter DE, and join DG; the chord DG is the mean proportional required.

For join GE. The triangle DGE, being contained in a semicircle, is right-angled, and therefore (VI. 16. cor. 1.) DG is a mean proportional between DF and DE, that is, between the given straight lines A and B.



Or thus.

Having made DF and DE equal to A and B, on FE describe the semicircle FGE; and the tangent DG being drawn, is the mean proportional required. For (III. 32. cor. 2.) DF$\times$DE$=$DG^2, and consequently (V. 6.) DF : DG : : DG : DE †.



---

* See Note XLIII. † See Note XLIV.

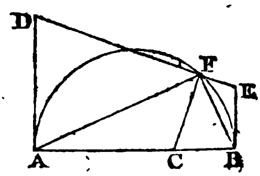
PROP. XIX. PROB.

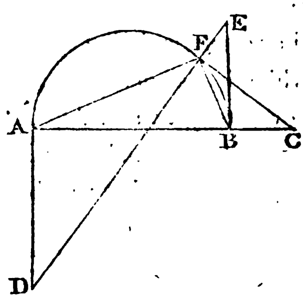
To divide a straight line, whether internally or externally, so that the rectangle under its segments shall be equivalent to a given rectangle.

Let AB be a straight line, which it is required to cut, so that the rectangle under its segments shall be equivalent to a given rectangle.

On AB describe the semicircle AFB, at A and B apply tangents AD and BE equal to the sides of the given rectangle, join DE, to which, and from the point F where it meets the circumference, draw the perpendicular FC; this will divide AB into the segments required.

For join AF and BF. And because AD is a tangent and AF a straight line inflected to the circumference, the exterior angle DAF is equal to CBF which stands in the alternate segment (III. 25. and III. 19. cor. 1.); and, for the same reason, the exterior angle EBF is equal to CAF. But the opposite angles DAC and DFC of the quadrilateral figure ADFC are, in the first case, two right angles, and therefore the angle ADF is (III. 19. cor. 1.) equal to BCF; and, in the second case, the angles DAC and DFC being both right angles, the figure DAFC is contained in a semicircle, consequently (III. 18.) the angle ADF is equal to BCF. In the same manner, it is proved in both cases, that the angle BEF is equal to ACF; wherefore the triangles DAF and AFC are similar to BCF and BFE; and

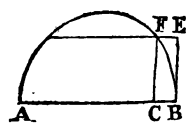




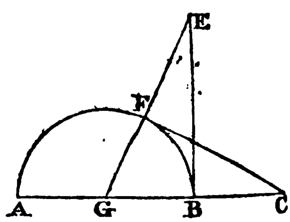
hence AD : AF : : CB : BF, and AF : AC : : BF : BE; consequently (V. 16.) AD : AC : : CB : BE, and (V. 6.) AD.BE = AC.CB.

Cor. If the sides of the given rectangle be equal, the construction of the problem will become materially simplified.

First, in the case of internal section: The tangents AD, BE being equal, it is evident that DE must be parallel to AB and the perpendicular FC parallel to EB. Whence, employing this construction, the rectangle under the segments AC and CB is equivalent to the square of BE; which also follows from Prop. 32. Book III.



Next, in the case of external section: The opposite tangents AD, BE being equal, the triangles AGD and BGE are evidently equal, and therefore DE passes through the centre. Hence the triangles BGE and FGC are also equal, and GC equal to GE.—This construction being effected, the rectangle AC, CB will be equal to the square of BE; which is also deduced from Prop. 32. Book III., since CF is now a tangent and AC.CB = CF^2 or BE^2.



If AB be equal to BE, the construction will exactly correspond with what was before given.

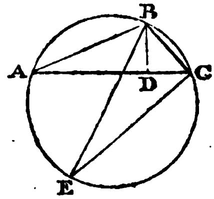
PROP. XX. THEOR.

The rectangle under any two sides of a triangle, is equivalent to the rectangle under the perpendicular drawn to the base and the diameter of the circumscribing circle.

Let ABC be a triangle, about which is described a circle

having the diameter BE; the rectangle under the sides AB and BC is equivalent to the rectangle under BE and the perpendicular BD let fall from the vertex of the triangle upon the base AC.

For join CE. And the angle BAD is equal to BEC (III. 18.), since they both stand upon the same arc BC; and the angle ADB, being a right angle, is equal to ECB, which is contained in a semicircle (III. 22.). Wherefore the triangles ABD and EBC, being thus similar (VI. 12.), AB : BD : : EB : BC, and consequently (V. 6.) AB.BC = EB.BD.



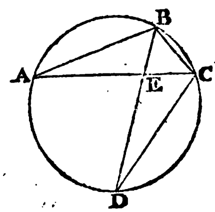
PROP. XXI. THEOR.

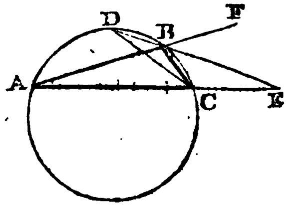
The square of a straight line that bisects, whether internally or externally, the vertical angle of a triangle, is equivalent to the difference between the rectangle under the sides, and the rectangle under the segments into which it divides the base.

In the triangle ABC, let BE bisect the vertical angle CBA or its adjacent angle CBF; then $BE^2 = AB.BC - AE.EC$, or $AE.EC - AB.BC$.

For (III. 10. cor.) about the triangle describe a circle, produce BE to the circumference, and join CD.

The angles BAE and BDC, standing upon the same arc BC, are (III. 18.) equal, and the angle ABE is, by hypothesis, equal to DBC; wherefore (VI. 12.) the triangles AEB and DCB are similar, and AB : BE : : DB : BC. Consequently (V. 6.) AB.BC = BE.BD;





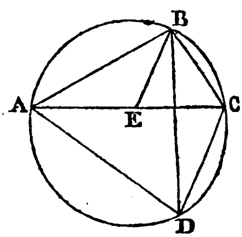
but $BE.BD = BE.ED + BE^2$, or $BE.ED - BE^2$, and (III. 32.) $BE.ED = AE.EC$; wherefore $AB.BC = BE.ED + BE^2$, or $BE.ED - BE^2$, and consequently $BE^2 = AB.BC - AE.EC$ or $AE.EC - AB.BC$ *.

PROP. XXII. THEOR.

The rectangles under the opposite sides of a quadrilateral figure inscribed in a circle, are together equivalent to the rectangle under its diagonals.

In the circle ABCD, let a quadrilateral figure be inscribed, and join the diagonals AC, BD; the rectangles AB, CD and BC, AD, are together equivalent to the rectangle AC, BD.

For (I. 4.) draw BE, making an angle ABE equal to CBD.



The triangles AEB and DCB having thus the angle ABE equal to DBC, and the angle BAE or BAC equal (III. 18.) to BDC, are similar (VI. 12.), and therefore AB : AE : : BD : CD; whence (V. 6.) $AB.CD = AE.BD$. Again, because the angle ABE is equal to DBC, add EBD to each, and the whole angle ABD is equal to EBC; and the angle ADB is equal to ECB (III. 18.); wherefore the triangles DAB and CEB are similar (VI. 12.), and AD : BD : : EC : BC, and consequently $BC.AD = EC.BD$. Whence the rectangles AB, CD and BC, AD are together equal to the rectangles AE, BD and EC, BD, that is, to the whole rectangle AC, BD†.

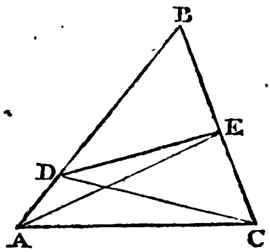
---

* See Note XLV. † *See Note XLVI.*

PROP. XXIII. THEOR.

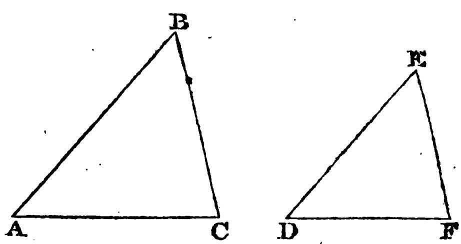
Triangles which have a common angle, are to each other in the compound ratio of the containing sides.

Let ABC and DBE be two triangles, having the same or an equal angle at B; ABC is to DBE in the ratio compounded of that of BA to BD, and of BC to BE.



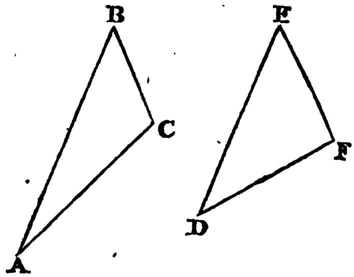
For join AE and CD. The ratio of the triangle ABC to DBE may be conceived as compounded of that of ABC to DBC, and of DBC to DBE. But (V. 25. cor. 2.) the triangle ABC is to DBC, as the base BA to BD; and, for the same reason, the triangle DBC is to DBE, as the base BC to BE; consequently the triangle ABC is to DBE in the ratio compounded of that of BA to BD, and of BC to BE, or (V. 23.) in the ratio of the rectangle under BA and BC to the rectangle under BD and BE.

Cor. 1. Hence similar triangles are in the duplicate ratio of their homologous sides. For, if the angle at B be equal to that at E, the triangle ABC is to DEF in the ratio compounded of that of AB to DE, and of CB to FE; but, these triangles being similar,



the ratio of AB to DE is the same as that of CB to FE (VI. 12.), and consequently the triangle ABC is to DEF in the duplicate ratio of AB to DE, or (V. 24.) as the square of AB to the square of DE.

Cor. 2. Hence triangles which have the sides that contain an equal angle reciprocally proportional, are equivalent. For, the angle at B being equal to that at E, the triangle ABC is to DEF, as AB.CB to DE.FE; but AB : DE :: FE : CB, and (V. 6.) AB.CB=DE.FE; consequently (V. 4.) the third and fourth terms of the analogy being equal, the first and second must also be equal.



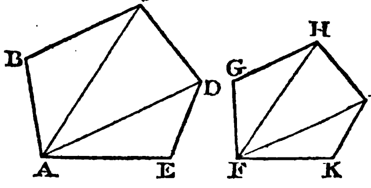
PROP. XXIV. THEOR.

Similar rectilineal figures may be divided into corresponding similar triangles.

Let ABCDE and FGHIK be similar rectilineal figures, of which A and F are corresponding points; these figures may be resolved into a like number of triangles respectively similar.

For, from the point A in the one figure, draw the straight lines AC, AD, and, from F in the other, draw FH, FI; the triangles BAC, CAD, and DAE are respectively similar to GFH, HFI, and IFK.

Because the polygon ABCDE is similar to FGHIK, the angle ABC is equal to FGH, and AB : BC :: FG : GH; wherefore (VI. 14.) the triangle BAC is similar to GFH. Hence the angle BCA is equal to GHF; and the whole angle BCD being equal to GHI, the remaining angle ACD must be equal



to FHI. But BC : AC : : GH : FH, and BC : CD : : GH : HI, consequently (V. 15.) AC : CD : : FH : HI, and the triangles CAD and HFI (VI. 14.) are similar. Whence the angle CDA being equal to HIF and the angle CDE to HIK, the angle ADE is equal to FIK; and since CD : DA : : HI : IF, and CD : DE : : HI : IK, therefore (V. 15.) DA : DE : : IF : IK, and the triangles DAE and IFK are similar.

The same train of reasoning, it is obvious, would apply to polygons of any number of sides.

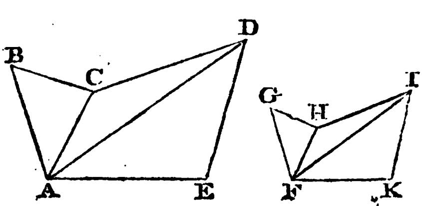
PROP. XXV. PROB.

On a given straight line, to construct a rectilineal figure similar to a given rectilineal figure.

Let FK be a straight line, on which it is required to construct a rectilineal figure similar to the figure ABCDE.

Join AC and AD, dividing the given rectilineal figure into its component triangles: From the points F and K draw FI and KI, making the angles KFI and FKI equal to EAD and AED; from F and I draw FH and IH making the angles IFH and FIH equal to DAC and ADC; and lastly from F and H draw FG and HG making the angles HFG and FHG equal to CAB and ACB. The figure FGHIK is similar to ABCDE.

For the several triangles KFI, IFH, and HFG, which compose the figure FGHIK, are, by the construction, evidently similar to the triangles EAD, DAC, and CAB, into which the figure ABCDE was resolved. Whence FK : KI : : AE : ED; also KI : IF : : ED : DA, and IF : IH : : DA : DC, and consequently (V. 15.) KI : IH : : ED : DC.



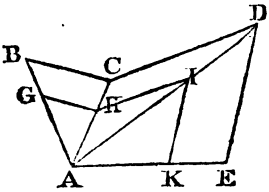
Again, IH : HF : : DC : CA, and HF : HG : :

CA : CB; and hence IH : HG :: DC : CB. But HG : GF :: CB : BA: and the ratio of GF to FK, being compounded of that of GF to FH, of FH to FI, and of FI to FK, is the same with the ratio of BA to AE, which is compounded of the like ratios of BA to AC, of AC to AD, and of AD to AE. Wherefore all the sides about the figure FGHIK are proportional to those about ABCDE; but the several angles of the former, having a like composition, are respectively equal to those of the latter. Whence the figure FGHIK is similar to the given figure.

The same reasoning, it is manifest, would extend to polygons of any number of sides.

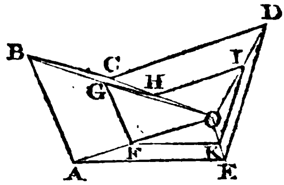
Scholium. The general solution of this problem is derived from the principle, that similar triangles, by their composition, form similar polygons. The mode of construction, however, admits of some variation. For instance, if the straight line FK be parallel to AE, or in the same extension with that homologous side,—the several triangles FIK, FHI, and FGH may be more easily constituted in succession, by drawing the straight lines FI and KI, FH and IH, and FG and GH parallel to the corresponding sides in the original figure ABCDE; because (I. 31.) a corresponding equality of angles will be thus produced.

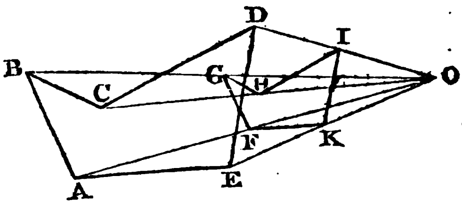
But, if FK have no determinate position, the construction may be still farther simplified; For, having made AK equal to that base and joined AD and AC, draw KI, IH, and HG parallel to ED, DC, and CB. The figure AKIHG is evidently similar to AEDCB, since its component triangles have the same vertical angles as those of the original figure, and (I. 23.) the angles at the base are equal.



If the given base FK be parallel to the corresponding side AE of the original figure, a more general construction will result. Join AF, EK and produce them to meet in O; join

OB, OC, and OD, and draw FG, GH, HI, and therefore IK, parallel to AB, BC, CD, and DE: The figure FGHIK thus formed is similar to ABCDE. For the triangles KOF, FOG, GOH, HOI, and IOK are evidently similar to the triangles EOA, AOB, BOC, COD, and DOE. But these triangles compose severally the two polygons, when the point O lies within the original figure; and when that point of concurrence lies without the figure ABCDE, the similar triangles IOK and DOE being taken away from the similar compound polygons FGHIOK and ABCDOE, there remains the figure FGHIK similar to the original one.





It farther appears, from these investigations, that a rectilineal figure may have its sides reduced or enlarged in a given ratio, by assuming any point O and cutting the diverging lines OE, OA, OB, OC, and OD in that ratio; the corresponding points of section being joined, will exhibit the figure required *.

PROP. XXVI. THEOR.

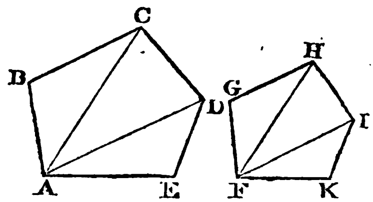
Of similar figures, the perimeters are proportional to the corresponding sides, and the areas are in the duplicate ratio of those homologous terms.

Let ABCDE and FGHIK be similar polygons, which have the corresponding sides AB and FG; the perimeter, or linear boundary, ABCDE is to the perimeter FGHIK, as AB to

* See Note XLVII.

FG, BC to GH, CD to HI, DE to IK, or EA to KF; but the area of ABCDE, or the contained surface, is to the area of FGHIK, in the duplicate ratio of AB to FG, of BC to GH, of CD to HI, of DE to IK, or of EA to KF.

For, by drawing the diagonals AC, AD in the one, and FH, IF in the other, these polygons will be resolved into similar triangles. Whence the several analogies AB : BC : : FG : GH, BC : AC : : GH : FH, AC : CD : : FH : HI, CD : AD : : HI : FI, and AD : DE : : FI : IK; wherefore, by equality and alternation, AB : FG : : BC : GH : : CD : HI : : DE : IK, and consequently (V. 19.) as one of the antecedents AB, BC, CD, DE or AE, is to its consequent FG, GH, HI, IK or FK, so is the amount of all those antecedents, or the perimeter ABCDE, to the amount of all the consequents, or the perimeter FGHIK.



Again, the triangle CAB is to the triangle HFG (VI. 23. cor. 1.) in the duplicate ratio of AB to FG,—the triangle DAC is to the triangle IFH in the duplicate ratio of AC to FH, or of AB to FG,—and the triangle EAD is to KFI in the duplicate ratio of AD to FI or of AB to FG; wherefore (V. 19.) the aggregate of the triangles CAB, DAC, and EAD, or the area of the polygon ABCDE, is to the aggregate of the triangles HFG, IFH, and KFI, or the area of the polygon FGHIK, in the duplicate ratio of AB to FG, of BC to GH, of CD to HI, or of DE to IK.

Cor. Hence also the perimeter ABCDE is to the perimeter FGHIK, as any diagonal AD to the corresponding diagonal FI, and the area ABCDE is to the area FGHIK in the duplicate ratio of AD to FI.

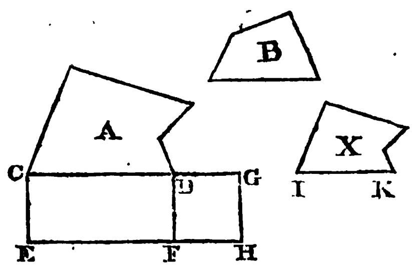
PROP. XXVII. PROB.

To construct a rectilineal figure that shall be similar

to one, and equivalent to another, given rectilineal figure.

Let it be required to describe a rectilineal figure similar to A, and equivalent to B.

On CD a side of A, describe (II. 9.) equivalent to that figure, the rectangle CDFE, and on DF describe the rectangle DGHF equivalent to the figure B, find (VI. 18.) IK a mean proportional between CD and DG, and on IK construct, in the same position, a figure X similar to the rectilineal figure A; this will be likewise equivalent to B.



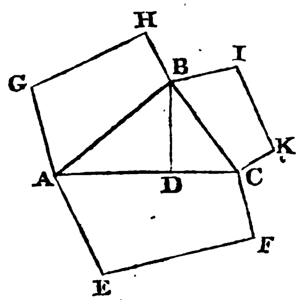
For the figures A and X, being similar, must (VI. 26.) be in the duplicate ratio of their homologous sides CD and IK; and since IK is a mean proportional between CD and DG, the duplicate ratio of CD to IK is the same as the ratio of CD to DG (V. 23.); consequently the figure A is to the figure X as CD to DG, or (V. 25.) as the rectangle CF to the rectangle DH; but the figure A is equivalent to the rectangle CF, and therefore (V. 4.) the figure X is equivalent to the rectangle DH, that is, to the figure B.

PROP. XXVIII. THEOR.

A rectilineal figure described on the hypotenuse of a right-angled triangle, is equivalent to similar figures described on the two sides.

Let ABC be a right-angled triangle; the figure ACFE described on the hypotenuse, is equivalent to the similar figures AGHB and BIKC, described on the sides AB and BC.

For draw BD perpendicular to the hypotenuse. And since (VI.16.cor.1.) AC : AB :: AB : AD, therefore AC is to AD in the duplicate ratio of AC to AB, that is, (VI. 26.), as the figure on AC to the figure on AB. For the same reason, AC is to CD in the duplicate ratio of AC to BC, or as the figure on AC to the figure on BC. Whence (V. 19. cor. 2.) AC is to the two segments AD and CD taken together, as the figure on AC to both the figures on AB and BC; and the first term of the analogy being thus equal to the second, the third must be equal to the fourth (V. 4.), or the figure described on the hypotenuse is equivalent to the similar figures described on the two sides.

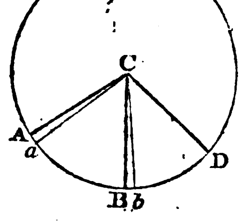


PROP. XXIX. THEOR.

The arcs of a circle are proportional to the angles which they subtend at the centre.

Let the radii CA, CB, and CD intercept arcs AB and BD; the arc AB is to BD, as the angle ACB to BCD.

For (I. 5.) bisect the angle ACB, bisect again each of its halves, and repeat the operation indefinitely. An angle AC*a* will be thus obtained less than any assignable angle. Let this angle AC*a* or BC*b* (I. 4.) be repeatedly applied about the point C, from BC towards DC; it must hence, by its multiplication, fill up the angle BCD, nearer than any possible difference. But the elementary angle AC*a* being equal to BC*b*, the corresponding arc A*a* is (III. 13.) equal to B*b*. Consequently this arc A*a* and its angle AC*a*, are like



measures of the arc AB and the angle ACB, and they are both contained equally in the arc BD and its corresponding angle BCD. Wherefore AB : BD :: ACB : BCD.

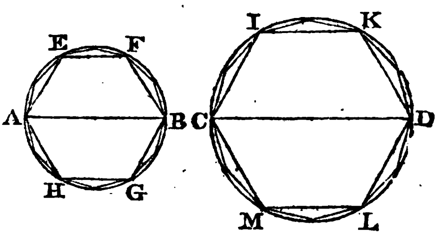
Cor. Hence the arc AB is also to BD, as the sector ACB to the sector BCD; for these sectors may be viewed as alike composed of the elementary sector ACa.

PROP. XXX. THEOR.

The circumference of a circle is proportional to the diameter, and its area to the square of that diameter.

Let AB and CD be the diameters of two circles;—the circumference AFG is to the circumference CKL, as AB to CD; and the area contained by AFG is to the area contained by CKL, as the square of AB to the square of CD.

For inscribe the regular hexagons AEFBGH and CIKDLM. Because these polygons are equilateral and equiangular, they are similar; and consequently (VI. 26. cor.) the diagonal AB is to the corresponding diagonal CD, as the perimeter AEFBGH to the perimeter CIKDLM. But this proportion must subsist, whatever be the number of chords inscribed in either semicircumference. Insert a dodecagon in each circle between the hexagon and the circumference, and its perimeter will evidently approach nearer to the length of that circumference. Proceeding thus, by repeated duplications, — the perimeters of the series of polygons that

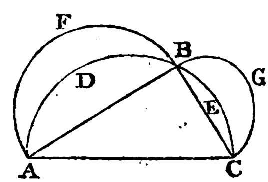


emerge in succession, will continually approximate to the curvilineal boundary, which forms their ultimate limit. Where-

fore this extreme term, or the circumference AEFBGH, is to the circumference CIKDLM, as the diameter AB to the diameter CD.

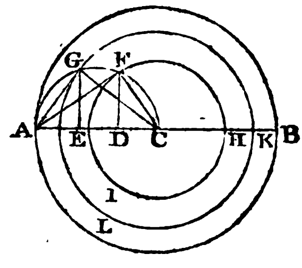
Again, the hexagon AEFBGH (VI. 26. cor.) is to the hexagon CIKDLM in the duplicate ratio of the diagonal AB to the corresponding diagonal CD, or (V. 24.) as the square of AB to the square of CD. Wherefore the successive polygons, which arise from a repeated bisection of the intermediate arcs, and which approach continually to the areas of their containing circles, must have still that same ratio. Consequently the limiting space, or the circle AEFBGH, is to the circle CIKDLM, as the square of AB to the square of CD.

Cor. 1. It hence follows, that if semicircles be described on the sides AB, BC of a right-angled triangle, and on the hypotenuse AC another semicircle be described, passing (III. 22.) through the vertex B, the crescents AFBD and BGCE are together equivalent to the triangle ABC. For, by the Proposition, the square of AC is to the square of AB, as the circle on AC to the circle on AB, or (V. 3.) as the semicircle ADBEC to the semicircle AFB; and, for the same reason, the square of AC is to the square of BC, as the semicircle ADBEC to the semicircle BGC. Whence (V. 20.) the square of AC is to the squares of AB and BC, as the semicircle ADBEC to the semicircles AFB and BGC. But (II. 11.) the square of AC is equivalent to the squares of AB and BC, and therefore (V. 4.) the semicircle ADBEC is equivalent to the two semicircles AFB and BGC; take away the common segments ADB and BEC, and there remains the triangle ABC equivalent to the two crescents or *lunes* AFBD and BGCE.



Cor. 2. Hence the method of dividing a circle into equal portions, by means of concentric circles. Let it be required, for instance, to trisect the circle of which AB is a diameter.

Divide the radius AC into three equal parts, from the points of section draw perpendiculars DF, EG meeting the circumference of a semicircle described on AC, join CF, CG, and from C as a centre, with the distances CF, CG, describe the circles FHI, GKL: The circle on AB will be divided into three equal portions, by those interior circles. For, join AF and AG: Because AFC, being in a semicircle, is a right angle (III. 22.), AC is to CD (VI. 16. cor. 1. and V. 24.), as the square of AC to the square of CF, that is, as the circle on AB to the circle FHI; but CD is the third part of AC; wherefore (V. 5.) the circle FHI is the third part of the circle on AB. In like manner, it is proved, that the circle GKL is two-third parts of the circle on AB. Consequently, the intervening annular spaces, and the circle FHI, are all equal*.



PROP. XXXI. THEOR.

The area of any triangle is a mean proportional between the rectangle under the semiperimeter and its excess above the base, and the rectangle under the separate excesses of that semiperimeter above the two remaining sides.

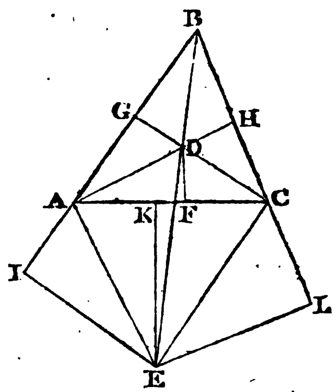
The area of the triangle ABC is a mean proportional between the rectangle under half the sum of all the sides and its excess above AC, and the rectangle under the excess of that semiperimeter above AB and its excess above BC.

For produce the sides BA and BC, draw the straight lines BE, AD, and AE bisecting the angles CBA, BAC, and CAI,

* See Note XLVIII.

and let fall the perpendiculars DF, DG, and DH within the triangle, and the perpendiculars EI, EK, and EL without it.

The triangles ADF and ADG, having the angle DAF equal to DAG, the angles F and G right angles, and the common side AD,—are (I. 21.) equal; for the same reason, the triangles BDG and BDH are equal. In like manner, it is proved, that the triangles AEI and AEK are equal, and the triangles BEI and BEL. Whence the triangles CDH and CDF, having the side DH equal to DF, the side DC common, and the right angle CHD equal to CFD,—are (I. 22.) equal; and, for the same reason, the triangles CEK and CEL are equal. The perimeter of the triangle ABC is therefore equal to twice the segments AF, FC, and BG; consequently BG is the excess of the semiperimeter above the base AC, and AG is the excess of that semiperimeter—or of the segments BH, HC, and AG,—above the side BC. But the sides AB and BC, with the segments AK and CK, or AI and CL, also form the perimeter; whence, BI being equal to BL, the part AI is the excess of the semiperimeter above the side AB.



Now, because DG and EI, being perpendicular to BI, are parallel, BG : DG : : BI : EI (VI. 2.), and consequently (V. 25. cor. 2.) BI × BG : BI × DG : : DG × BI : DG × EI. But since AD and AE bisect the angle BAC and its adjacent angle CAI, the angles GAD and EAI are together equal to a right angle, and equal, therefore, to IEA and EAI; whence the angle GAD is equal to IEA, and the right-angled triangles DGA and AIE are similar. Wherefore (VI. 12.) DG : AG : : AI : EI, and (V. 6.) DG × EI = AG × AI; consequently BI × BG : DG × BI : : DG × BI : AG × AI. But the tri-

angle ABC is composed of three triangles ADB, BDC, and CDA, which have the same altitude; and therefore its area is equal to the rectangle under DG and half their bases AB, BC, and AC, or the semiperimeter BI. Whence the area of the triangle ABC is a mean proportional between the rectangle under BI and its excess above AC, and the rectangle under its excess above BC and that above AB.

Cor. If the area of a triangle be expressed by A, its sides by a, b, and c, and the semiperimeter by s; then $s(s-a) : A :: A : (s-b)(s-c)$, and consequently $A^2 = s(s-a)(s-b)(s-c)$, and $A = \sqrt{\left(s(s-a)(s-b)(s-c)\right)}$ *.

PROP. XXXII. PROB.

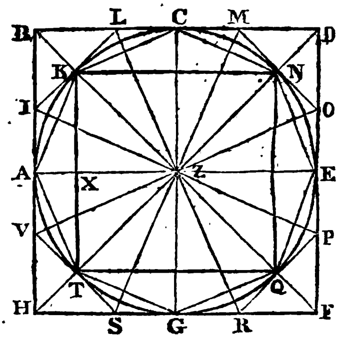
Given the area of an inscribed, and that of a circumscribed, regular polygon; to find the areas of inscribed and circumscribed regular polygons, having double the number of sides.

Let TKNQ and HBDF be given similar inscribed and circumscribed rectilineal figures; it is required thence to determine the surfaces of the corresponding inscribed and circumscribed polygons AKCNEQGT and VILMOPRS, which have twice the number of sides.

From the centre of the circle, draw radiating lines to all the angular points. It is evident that the triangles ZXK and ZAB are like portions of the given inscribed and circumscribed figures TKNQ and HBDF; and that the triangle ZAK, and the quadrilateral figure ZAIK are also like portions of the derivative polygons AKCNEQGT and VILMOPRS. And since XK is parallel to AB, ZX : ZA :: ZK : ZB (VI. 2.); but ZX is to ZA as the triangle ZXK is to the tri-

* See Note XLIX.

angle ZAK (V. 25. cor. 2.), and, for the same reason, ZK is to ZB as the triangle ZAK is to the triangle ZAB; whence ZXK : ZAK : : ZAK : ZAB, and consequently the derivative inscribed polygon AKCNEQGT is a mean proportional between the inscribed and circumscribed figures TKNQ and HBDF.



Again, because ZI bisects the angle AZB, ZA is to ZB, or ZX is to ZK, as AI to IB (VI. 11.), and consequently (V. 25. cor. 2.) the triangle XZK is to the triangle AZK, as the triangle AZI to the triangle IZB. Hence the inscribed figure TKNQ is to its derivative inscribed figure AKCNEQGT as the triangle AZI to the triangle IZB; wherefore (V. 11. and 13.) TKNQ and AKCNEQGT together are to twice TKNQ, as the triangles AZI and IZB, or AZB, to twice the triangle AZI, or the space AIKZ,—that is, as HBDF to VILMOPRS. And thus the two inscribed polygons are to twice the simple inscribed polygon, as the surface of the circumscribing polygon, to the surface of the derivative circumscribing polygon with double the number of sides.

Cor. Hence the area of a circle is equivalent to the rectangle under its radius and a straight line equal to half its circumference. For the surface of any regular circumscribing polygon, such as VILMOPRS, being composed of a number of triangles AZI, which have all the same altitude ZA, is equivalent (II. 6.) to the rectangle under ZA and half the sum of their bases, or the semiperimeter of the polygon. But the circle itself, as it forms the ultimate limit of the polygon, must have its area, therefore, equivalent to the rectangle under the radius ZA, and the semicircumference ACE.

Scholium. This Proposition furnishes the best elementary method of approximating to the numerical expression for the area of a circle. Suppose the radius of a circle to be denoted by unit: The surface of the circumscribing square will be expressed by 4, and consequently (IV. 16. cor.) that of its inscribed square by 2. Wherefore the surface of the inscribed octagon is $=\sqrt{2 \times 4}=2{,}8284271$; and the surface of the circumscribing octagon is found by the analogy, $2+2{,}8284271 : 2 \times 2 :: 4 : 3{,}3137085$. Again, $\sqrt{(2{,}8284271 \times 3{,}3137085)} = 3{,}0614674$, which expresses the area of the inscribed polygon of 16 sides; and $2{,}8284271+3{,}0614674 : 2 \times 2{,}8284271$, or $5{,}8898945 : 5{,}6568542 :: 3{,}3137085 : 3{,}1825979$, which denotes the area of the circumscribing polygon of 16 sides. Pursuing this mode of calculation, by alternately extracting a square root and finding a fourth proportional, the following Table will be formed, in which the numbers expressing the surfaces of the inscribed and circumscribed polygons continually approach to each other, and consequently to the measure of their intermediate circle.

Number of Sides.	Area of the inscribed Polygon.	Area of the circumscribing Polygon.
4	2,0000000	4,0000000
8	2,8284271	3,3137085
16	3,0614674	3,1825979
32	3,1214451	3,1517249
64	3,1365485	3,1441184
128	3,1403311	3,1422236
256	3,1412772	3,1417504
512	3,1415138	3,1416321
1024	3,1415729	3,1416025
2048	3,1415877	3,1415951
4096	3,1415914	3,1415933
8192	3,1415923	3,1415928
16384	3,1415925	3,1415927
32768	3,1415926	3,1415926

Hence 3,1415926 is the nearest expression, consisting of seven decimal places, for the area of a circle whose radius is 1. But the semicircumference in this case denoting also the surface, the same number must represent the circumference of a circle whose diameter is 1. Consequently, if D denote the diameter of any circle, the circumference will be expressed approximately, by $3,1415926 \times D$; whence the area will be $\frac{1}{4}D^2 \times 3,1415926$, or $D^2 \times ,78539815$ *.

Since the four last decimals 5926 come so near to 6000, it will, in most cases, be sufficiently accurate to reckon the circumference equal to $D \times 3,1416$, and its area equal to $D^2 \times ,7854$. But other approximations, expressed in lower numbers, may be found, by help of Prop. 28. Book V. For $m=3$, $n=7$, $p=16$, and $q=11$; whence, remounting successively from these conditional equalities, the ratio of the diameter to the circumference of a circle is denoted in progression, by 1 : 3—by 7 : 22—by 113 : 355—and by 1250 : 3927. Hence also the circle is to its circumscribing square nearly—as 11 to 14, or still more nearly—as 355 to 452.

* See Note L.

APPENDIX.

The constructions used in Elementary Geometry, were effected, by the combination of straight lines and circles. Many problems, however, can be resolved, by the single application of the straight line or the circle; and such solutions are not only interesting, from the ingenuity and resources which they display, but may, in a variety of instances, be employed with manifest advantage. This Appendix is intended to exhibit a selection of Geometrical Problems, resolved by either of those methods singly. It is accordingly divided into Two Parts, corresponding to the rectilineal and the circular constructions.

PART I.

Problems resolved by help of the Ruler, or by Straight Lines only.

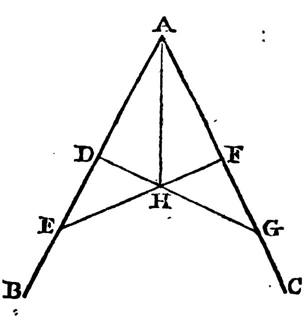
PROP. I. PROB.

To bisect a given angle.

Let BAC be an angle, which it is required to bisect, by drawing only straight lines.

In AB take any two points D and E, from AC cut off AF equal to AD and AG to AE, draw EF and DG, crossing in the point H: AH will bisect the angle BAC.

For the triangles EAF and DAG, having the sides EA and AF equal by construction to GA and AD, and the contained angle DAG common to both, are equal (I. 3.), and consequently the angle AEF is equal to AGD. And since AE is equal to AG, and the part AD to AF, the remainder DE must be equal to FG; wherefore the triangles DEH and HGF, having the angle at E equal to that at G, the vertical angles at H equal, and also their opposite sides DE and FG, are equal (I. 21.); and hence the side DH is equal to FH. Again, the sides AD and DH are equal to AF and FH, and AH is common to the two triangles AHD and AHF, which are therefore equal (I. 2.), and consequently the angle DAH is equal to FAH.

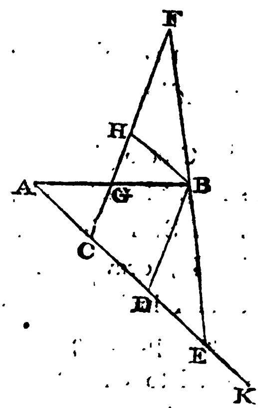


PROP. II. PROB.

To bisect a given finite straight line.

Let it be required to bisect AB, by a rectilineal construction.

Draw AK diverging from AB, and make AC=CD=DE, join EB and continue it beyond B till BF be equal to BE, and lastly join FC; which will bisect AB in the point G.



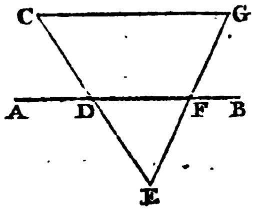
For draw BH parallel to AE. And because BD evidently bisects the sides EC and EF of the triangle CEF, it is parallel to the base CF (II. 4.); wherefore BDCH is a parallelogram, which has (I. 27.) its opposite sides BH and CD equal. But AC being parallel to BH, the angles GAC and GCA are equal to GBH and GHB, and the side AC, being made equal to CD, is hence equal to its corresponding interjacent side BH; whence the triangles AGC and BGH are equal (I. 21.), and therefore AG is equal to BG.

PROP. III. PROB.

Through a given point, to draw a line parallel to a given straight line.

Let it be required, by a rectilineal construction, to draw through C a parallel to AB.

In AB take any two points D and F, join CD, which produce till DE be equal to it; again join E with the point F, and continue this till FG be equal to EF: Then CG, being joined, will be parallel to AB.



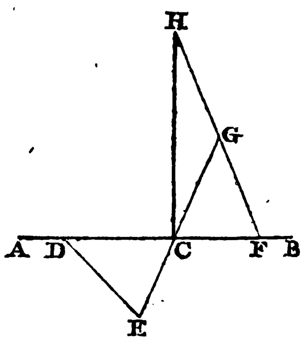
For, since AB or DF evidently bisects the sides EC and EG of the triangle CEG, it must be parallel to the base CG (II. 4.).

PROP. IV. PROB.

From a point in a given straight line, to erect a perpendicular.

Let C be a given point, from which it is required, by help of straight lines merely, to erect a perpendicular to AB.

In AB, having taken any point D, draw DE equal to DC and inclined to AB, join EC and produce it until CG be equal to CD or DE, make CF equal to CE, join FG and produce this till GH be equal to GC: Then CH will be perpendicular to AB.



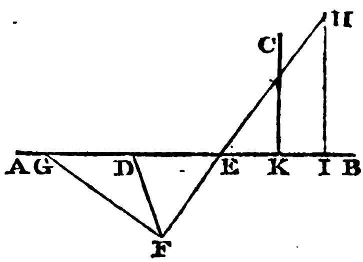
For the triangles DCE and GCF, having the sides DC, CE equal to GC, CF, and the contained angles vertical at C, are equal (I. 3.); whence FG=CD=CG=GH. The point G is therefore the centre of a semicircle which would pass through F, C, H, and consequently the angle FCH is a right angle (III. 22.), or CH is perpendicular to AB.

PROP. V. PROB.

To let fall a perpendicular upon a given straight line, from a point without it.

Let C be a given point, from which it is required, by a rectilineal construction, to let fall a perpendicular to AB.

In AB take any point D, draw DF obliquely, and make DE=DF=DG, join FE and produce it until EH be equal to EG, make EI=EF, join HI, and (Appendix, Part I. Prop. 3.) draw CK parallel to it: CK is the perpendicular required.



For the point D being obviously the centre of a semicircle passing through G, F, and E, the angle GFE is a right angle; and the triangles EGF, EHI, having the sides GE, EF equal to HE, EI, and their contained angles vertical,—are equal (I. 3.), and consequently the angle HIE is equal to GFE, or is a right angle; but since CK and HI are parallel, the angle CKA is equal to HIE (I. 23.), and therefore is also a right angle, or CK is perpendicular to AB.

PART II.

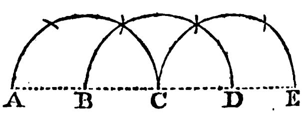
Geometrical Problems resolved by means of Compasses, or by the mere description of Circles.

PROP. I. PROB.

To repeat a given distance in the same direction.

Let A and B be two given points; it is required to find, by means of compasses only, a series of equidistant points in the same extended line.

From B as a centre, with the given distance BA, describe a portion of a circle, in which inflect that distance three times to C; from C, with the same radius, describe another circle, and insert the triple chords to D; repeat that process from D, E, &c.: The equidistant points A, B, C, D, E, &c. will all lie in the same straight line.



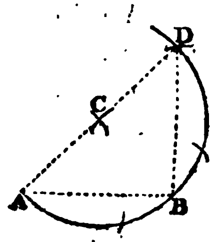
For, by this construction, three equilateral triangles are formed about the point B, and consequently (I. 32. cor. 1.) the whole angle ABC, made by the opposite distances BA and BC, is equal to two right angles, or ABC is a straight line. The same reason applies to the successive points, D, E, &c.

PROP. II. PROB.

To find the direction of a perpendicular from a given point to the straight line joining another given point.

Given the points A and B; to find a third point, such that the straight line connecting it with B shall be at right angles to BA.

From A and B, with any convenient distance, describe two arcs intersecting in C, from which, with the same radius, describe a portion of a circle passing through the points A and B, and insert that radius three times from A to D: BD is perpendicular to BA.



For it is evident, from the last Proposition, that the arc ABD is a semicircumference, and consequently that the angle ABD contained in it is a right angle.

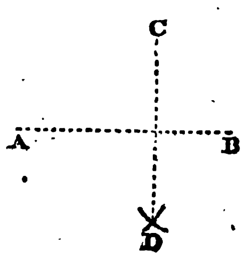
The construction would be somewhat simplified, by taking the distance AB for the radius.

PROP. III. PROB.

To find the direction of a perpendicular let fall from a given point upon the straight line which connects two given points.

Let C be a point, from which a perpendicular is to be let fall upon the straight line joining A and B.

From A as a centre, with the distance AC, describe an arc, and from B as a centre, with the distance BC, describe another arc, intersecting the former in the point D: CD is perpendicular to AB.



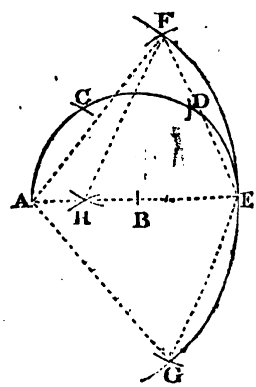
For CAD and CBD are evidently isosceles triangles, and consequently (I. 7.) their vertices must lie in a straight line AB which bisects their base CD at right angles.

PROP. IV. PROB.

To bisect a given distance.

Let A and B be two given points; it is required to find the middle point in the same direction.

From B as a centre, with the radius BA, describe a semicircle, by inserting that distance successively from A to C, D, and E; from A as a centre, with the distance AE, describe a portion of a circle FEG, in which, and from E, inflect the chords EF and EG equal to EC; and from the points F and G, with the same radius EC describe arcs intersecting in H: This point bisects the distance AB.



For, by the first Proposition, the points A, B, and E extend in a straight line; but the triangles FAG, FHG, and FEG, being evidently isosceles, their vertices A, H, and E (I. 7.) must lie in a straight line; whence the point H lies in the direction AB. Again, because EFH is an isosceles triangle, AF^2-HF^2 (II. 23. cor.) $=EA.AH$; that is, AE^2-EC^2 or (IV. 20. cor. 2.) $AB^2=EA.AH$. Wherefore, since EA is double of AB, the segment AH must be its half.

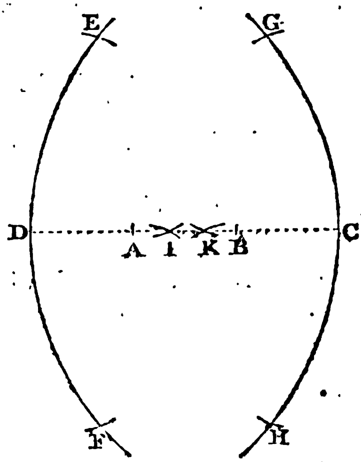
PROP. V. PROB.

To trisect a given distance.

Let it be required to find two intermediate points that are situate at equal intervals in the line of communication AB.

Repeat (App. II. 1.) the distance AB on both sides to C and D, from these points, with the radius CD, describe the

arcs EDF and GCH, from D and C inflect the chords DE and DF, CG and CH, all equal to DB, and, with the same distance and from the points E and F, G and H, describe arcs intersecting in I and K: The distance AB is trisected by points I and K.



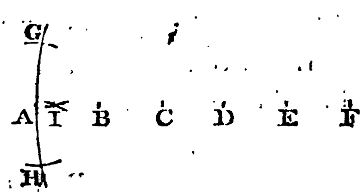
For it may be demonstrated, as in the last proposition, that the points I and K lie in the same direction AB. In like manner, it appears (II. 23. cor.) that $DG^2-KG^2=CD.DK$, or $9AB^2-4AB^2$ or $5AB^2=3AB.DK$; and consequently $5AB=3DK$, or $2AB=3AK$, and $AB=3BK$. But, for the same reason, $AB=3AI$.

PROP. VI. PROB.

To cut off any aliquot part of a given distance.

Suppose it were required to cut off the fifth part of the distance between the points A and B.

Repeat (App. II. 1.) the distance AB four times, to F; from F, with the radius FA, describe the arc GAH; inflect the chords AG and AH equal to AB, and, with that radius and from the points G and H, describe arcs intersecting in I: AI is the fifth part of the line of communication AB.



For, as before, the point I is situate in AB. But since

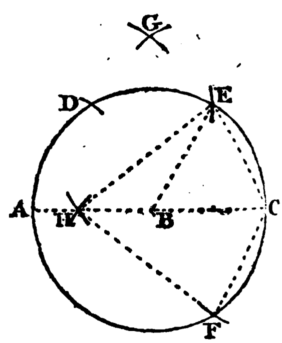
AGI is evidently an isosceles triangle and AF is equal to FG, it follows (II. 23. cor.) that $AG^2 = AF.AI$, and consequently $AB^2 = 5AB.AI$; whence $AB = 5AI$.

PROP. VII. PROB.

To divide a given distance by medial section.

Let it be required to cut the distance AB, such that $BH^2 = BA.AH$.

From B describe a circle with the radius BA, which insert successively from A to D, E, C, and F; from the extremities of the diameter AC and with the double chord AE, describe two arcs intersecting in G; and, from the points E and F with the distance BG, describe other two arcs intersecting in H: This is the point of medial section.



For it is evident that this point H lies in the straight line AB. And because the triangles AGB, CGB have their sides respectively equal, the angle ABG (I. 2.) is a right angle, and consequently (II. 11.) $AG^2 = AB^2 + BG^2$; but $AG = AE$, and $AE^2 = 3AB^2$ (IV. 20. cor. 2.); wherefore $3AB^2 = AB^2 + BG^2$, and $BG^2 = 2AB^2$. Now since $BE = EC$, it follows (II. 23. cor.) that $HE^2 - BE^2 = CH.HB$; but $HE^2 - BE^2 = BG^2 - BE^2 = AB^2$, and therefore $AB^2 = CH.HB$. Whence CH is cut by a medial section at B, and consequently (II. 22. cor. 1.) its greater segment BC or AB is likewise divided medially at H by the remaining portion BH.

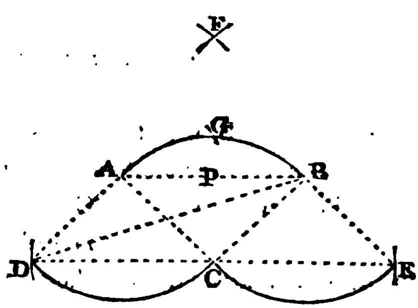
PROP. VIII. PROB.

To bisect a given arc of a circle.

Let it be required to bisect the arc AB of a circle whose centre is C.

From the extremities A and B, with the radius AC, describe opposite arcs, and from the centre C inflect the chord AB to D and E; from these points, with the distance DB describe arcs intersecting in F; and from D or E, with the distance CF, cut the given arc AB in G: AB is bisected in that point.

For the figures ABCD and ABEC being evidently rhomboids, DC and CE are parallel to AB, and hence constitute one straight line; consequently the triangles DFC and EFC having their corresponding sides equal, the angle DCF is a right angle, and (II. 11.) $DF^2 = DC^2 + CF^2$. But, in the rhomboid ABCD, $DB^2 + CA^2 = 2DC^2 + 2CB^2$ (II. 28.), or $DB^2 = 2DC^2 + CB^2$; and since $DB = DF$, $2DC^2 + CB^2 = DC^2 + CF^2$, whence $DC^2 + CB^2 = CF^2$, or $DC^2 + CG^2 = DG^2$, and therefore (II. 12.) DCG is a right angle. And because CG is perpendicular to DC, it is likewise (I. 23.) perpendicular to AB, and the triangles CAP and CBP are equal (I. 22.) and the angle ACG equal to BCG; whence (III. 13.) the arc AG = BG.

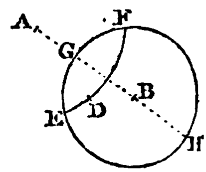


PROP. IX. PROB.

Given two points, to find the intersection of their connecting line with a given circumference.

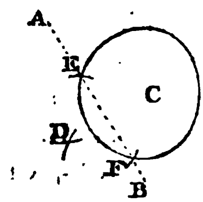
1. *Let one of the points be the centre of the circle.*

Take any point D within the circle, and from A, with the distance AD describe an arc cutting the circumference in E and F, bisect the arc EGF in G (App. II. 8.), and determine the semicircle GEH (App. II. 1.): G and H are the points of intersection of the straight line AGH.

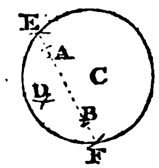


For the triangles AEB and AFB have their sides respectively equal, and consequently the angle ABE is equal to ABF (I. 2.); wherefore (III. 13.) the arc EG is equal to GF, or the straight line AH must bisect the arc EF.

2. *Let neither point lie in the centre of the circle.*



From A and B, with the distances AC and BC, describe arcs intersecting in D, from which, with the radius CE, cut the circumference in E and F: The straight line AB would extend through these points.



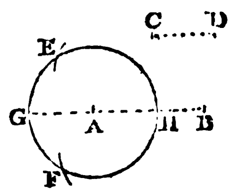
For the triangles CAD and CBD being isosceles, it appears from Book I. Prop. 7., that their vertices A and B lie in a perpendicular passing through the middle of the common base CD, and consequently the points E and F, which are vertices of the isosceles triangles CED and CFD, must likewise occur in the same straight line.

PROP. X. PROB.

To find the sum or difference of two given distances.

Let AB and CD be two distances, of which it is required to determine the sum and the difference.

From A with the distance CD describe a circle, cut the circumference in E and F by any arc described from B, bisect the arc EF (App. II. 8.) on both sides at G and H; BG will be the sum of the two distances, and BH their difference.



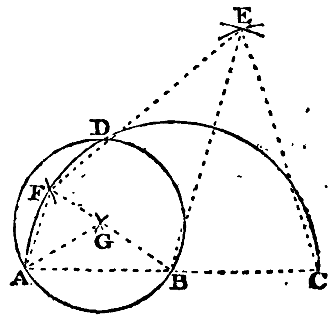
For GB, bisecting the chord EF at right angles, must pass through the centre A, and consequently the radius AG or CD is, on either side, added or taken away from AB.

PROP. XI. PROB.

To find the centre of a circle.

Assume an arc AB greater than a quadrant, and from one extremity B, with the distance BA, describe a semicircle ADC, cutting the given circumference in D; from the points B and C, with the distance CD, describe arcs intersecting in E, and, from that point with the same distance, describe an arc cutting ADC in F; and lastly, from the points A and B, with the distance AF, describe arcs intersecting in G: This point is the centre of the circle ADB.

For the isosceles triangles BEC, BEF, being evidently equal, the angle FBC is equal to both the angles at the base; but FBC is (I. 32. El.) equal to the interior angles BAF and BFA of the isosceles triangle ABF, and hence that triangle is similar to BEF. Wherefore BE : BF : : BA : AF, or CD : BD : : BA : AG; consequently the isosceles triangles CBD and BGA are similar, and the angle BCD is equal to GBA; BG is, therefore, parallel to CD, and hence (I. 32. El.) the angle BDC, or BCD, is equal to GBD. The



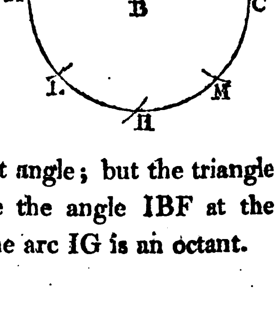
triangles BGA and BGD, having thus the side BA equal to BD, BG common, and equal contained angles GBA and GBD, are (I. 3. El.) equal, and therefore the side GA is equal to GD. The point G, being thus equidistant from three points, A, D, and B in the circumference, is hence (III. 8. cor.) the centre of the circle.

PROP. XII. PROB.

To divide the circumference of a given circle successively into four, eight, twelve, and twenty-four equal parts.

1. Insert the radius AB three times from A to D, E, and C; from the extremities of the diameter AC, and with a distance equal to the double chord AE, describe arcs intersecting in the point F; and from A, with the distance BF, cut the circumference on opposite sides at G and H: AG, GC, CH, and HA are quadrants.

For, as before, $AF^2 = AE^2 = 3AB^2$; and the triangle ABF being right-angled, $3AB^2 = AF^2 = AB^2 + BF^2$, and therefore $BF^2 = AG^2 = 2AB^2$; whence (II. 12.) ABG is a right angle, and AG a quadrant.



2. From the point F with the radius AB, cut the circle in I and K, and from A and C inflect the chord AI to L and M; the circumference is divided into eight equal portions by the points A, I, G, K, C, M, H, and L.

For BF^2, being equal to $2AB^2$, is equal to the squares of BI and IF, and consequently BIF is a right angle; but the triangle BIF is also isosceles, and therefore the angle IBF at the base is half a right angle; whence the arc IG is an octant.

3. The arc DG, on being repeated, will form twelve equal sections of the circumference.

For the arc AD is the sixth or two-twelfth parts of the circumference, and AG is the fourth or three-twelfths; consequently the difference DG is one-twelfth.

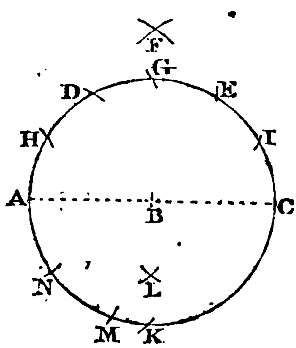
4. The arc ID is the twenty-fourth part of the circumference.

For the octant AI is equal to three twenty-fourths, and the sextant AD is equal to four twenty-fourths; their difference ID is hence one twenty-fourth part of the circumference.

PROP. XIII. PROB.

To divide the circumference of a given circle successively into five, ten, and twenty equal parts.

Mark out the semicircumference ADEC, by the triple insertion of the radius, from A and C, with the double chord AE, describe arcs intersecting in F, from A, with the distance BF, cut the circle in G and K, inflect the chords GH and GI equal to the radius AB, and, from the points H and I, with distance BF or AG, describe arcs intersecting in L.



It is evident from App. II. 7, that BL is the greater segment of the radius BH divided by a medial section; wherefore (IV. 22. cor. 2. El.) AL is equal to the side of the inscribed pentagon, and BL, to that of the decagon inscribed in the given circle. Hence AL may be inflected five times in the circumference, and BL ten times; and consequently the arc MK, or the excess of the fourth

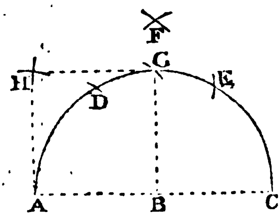
above the fifth, is equal to the twentieth part of the whole circumference.

PROP. XIV. PROB.

From a given side to trace out a square.

Let the points A and B terminate the side of a square, which it is required to trace.

From B as a centre describe the semicircle ADEC, from A and C, with the distance AE, describe arcs intersecting in F, from A, with the distance BF, cut the circumference in G, and from A and G, with the radius AB, describe arcs intersecting in H: The points H and G are corners of the required square.

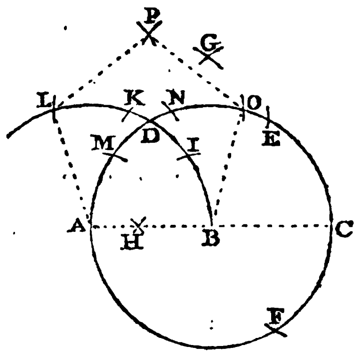


For (App. II. 12.) the angle ABG is a right angle, and the distances AB, AH, HG, and GB, are, by construction, all equal.

PROP. XV. PROB.

Given the side of a regular pentagon, to find the traces of the figure.

From B describe through A the circle ADECF, in which the radius is inflected four times, from A and C with the double chord AE describe arcs intersecting in G, from E and F, with the distance BG, describe arcs intersecting in H, from A, with the radius AB, describe a portion of a circle, inflect BH thrice from B to L and from A to O, and lastly from L and O, with the radius AB, describe arcs intersecting in P: The points A, L, P, O, B mark out the polygon.



For, from App. II. 7, it is evident that BH is the greater segment of the distance AB divided by a medial section. Consequently (IV. 3. El.) the isosceles triangles BAI, IAK, KAL, ABM, MBN, and NBO, have each of the angles at the base double their vertical angle. Wherefore the angles BAL and ABO are each of them six-fifths of a right angle (IV. 4. cor.), and hence (I. 33. cor.) the points L and O are corners of the pentagon; but P is evidently the vertex of the pentagon, since the sides LP and OP are each equal to AB.

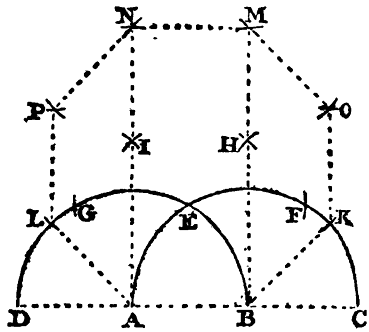
Scholium. The pentagon might also have been traced, as in Book IV. Prop. 5, by describing arcs from A and B with the distance HC, and again, from their intersection P, and with the radius AB, cutting those arcs in L and O. It is likewise evident, from Book IV. Prop. 8, that the same previous construction would serve for describing a decagon, P being made the centre of a circle in which AB is inflected ten times.

PROP. XVI. PROB.

The side of a regular octagon being given, to mark out the figure.

Let the side of an octagon terminate in the points A and B; to find the remaining corners of the figure.

On AB describe the two semicircles AEFC and BEGD; with the double chord AF, and from A, C and B, D describe arcs intersecting in H, I; from these points, with the radius AB, cut the semicircles in K, L: on HI describe the square HMNI, by making the diagonals HN, IM equal to BH, and



the sides equal to AB; and, on MH and NI, describe the rhombusses MOKH and NPLI: The points A, B, K, O, M, N, P, and L, are the several corners of the octagon.

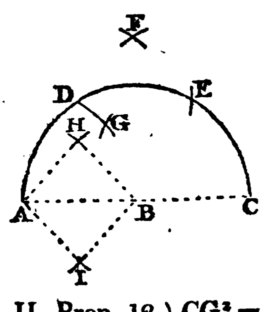
For (by App. II. Prop. 12.) BH, AI are both of them perpendicular to BA, and BKH, ALI are right angled isosceles triangles; HI is therefore parallel to BA, and HMNI, consisting of triangles equal to BKH, is a square; whence all the sides AB, BK, KO, OM, MN, NP, PL, and LA of the octagon are equal: But they likewise contain equal angles; for ABK, composed of ABH and HBK, is equal to three half right angles, and BKO, by reason of the parallels BH and KO, being the supplement of HBK, is also equal to three half right angles. In the same manner, the other angles of the figure may be proved to be equal.

PROP. XVII. PROB.

On a given diagonal to describe a square.

Let the points A and B be the opposite corners of a square which it is required to trace.

From B as a centre describe the semicircle ADEC, from A and C with the double chord AE describe arcs intersecting in F, from C with the distance BF describe an arc and cut this from A with the radius AD in G, and lastly from B and A with the distance BG describe arcs intersecting in H and I: ABHI is the required square.



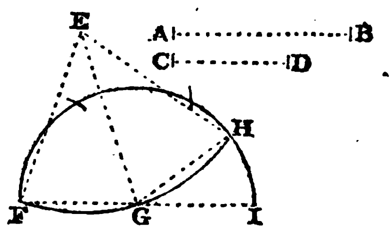
For, in the triangle AGC, the straight line GB bisects the base, and consequently (II. 25.) $AG^2 + CG^2 = 2AB^2 + 2BG^2$; but, (by App. II. Prop. 12,) $CG^2 =$

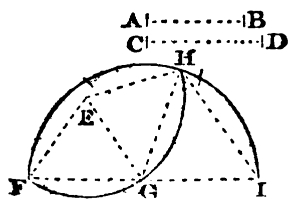
$BF^2 = 2AB^2$; whence $AG^2 = AB^2 = 2BG^2$, and (II. 12.) AHB is a right angle; and the sides AH, HB, BI, and IA being all equal, the figure is therefore a square.

PROP. XVIII. PROB.

Two distances being given, to find a third proportional.

Let it be required to find a third proportional to the distances AB and CD.





From any point E, and with the distance AB, describe a portion of a circle, in which inflect FG equal to CD, and from G, with that distance, describe the semicircle FHI; HI is the third proportional required.

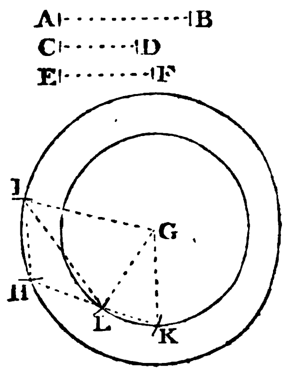
For the angles GEH and IGH are each of them double the angle GFH or IFH at the circumference (III. 17. El.); whence the triangles GEH and IGH must also have the angles at the base equal, and are consequently similar: Wherefore (VI. 12. El.) EG : GH : : GH : HI.

If the first term AB be less than half the second term CD, this construction, without some help, would evidently not succeed. But AB may be previously doubled, or assumed 4, 8, or 16 times greater, so that the circle FGH shall always cut FHI; and in that case, HI, being likewise doubled, or taken 4, 8, or 16 times greater, will give the true result.

PROP. XIX. PROB.

To find a fourth proportional to three given distances.

Let it be required to find a fourth proportional to the distances AB, CD, and EF.



From any point G, describe two concentric circles HI and KL with the distances AB and EF, in the circumference of the first inflect HI equal to CD, assume any point K in the second circumference, and cut this in L by an arc described from I with the distance HK; the chord LK is the fourth proportional required.

For the triangles ILG and HKG are equal, since their corresponding sides are evidently equal; whence the angle IGL is equal to HGK, and taking away HGL, the angle IGH remains equal to LGK; consequently the isosceles triangles GIH and GLK are similar, and GI : IH : : GL : LK, that is, AB : CD : : EF : LK.

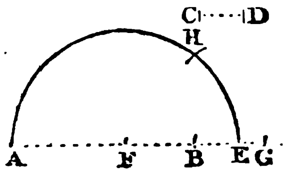
If the third term EF be more than double the first AB, this construction, it is obvious, will not answer without some modification. It may, however, be made to suit all the variety of cases, by multiplying equally AB and the chord LK, as in the last proposition.

PROP. XX. PROB.

To find a mean proportional between two given distances.

Let AB and CD be the two distances. To AB add

(App. II. 10.) BE equal to CD, bisect (App. II. 4.) AE in F, make BG equal to FB, from F describe the semicircumference AHE, and, with the same radius FE and from G as a centre, intersect it in H; BH is the mean proportional required.

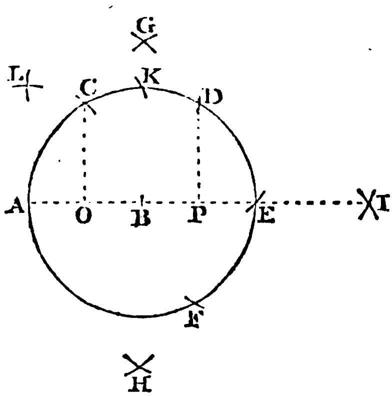


For (I. 5. cor.) it is evident, that BH is perpendicular to AE, and (III. 32. cor.), that $BH^2 = AB.BE$; whence (V. 6.) $AB : BH :: BH : BE$, or CD.

PROP. XXI. PROB.

To find the linear expressions for the square roots of the natural numbers, from one to ten inclusive.

This problem is evidently the same as, to find the sides of squares which are equivalent to the successive multiples of the square constructed on the straight line representing the unit. Let AB, therefore, be that measure: And from B as a centre, describe a circle, in which inflect the radius four times, from A to C, D, E, and F; from the opposite points A and E, with the double chord AD, describe arcs intersecting in G and H,—with the same distance, and from the points D, F, describe arcs intersecting in I,—and, with still the same distance and from E, cut the circumference in K; and from A and K, with the radius AB, describe arcs intersecting in L: Then will $AK^2 = 2AB^2$, $AD^2 = 3AB^2$, $AE^2 = 4AB^2$, $IK^2 = 5AB^2$, $IG^2 = 6AB^2$, $IC^2 = 7AB^2$, $GH^2 = 8AB^2$, $IA^2 = 9AB^2$, and $IL^2 = 10AB^2$.



For, in the isosceles triangles ACB and BDE, the perpen-

diculars CO and DP must bisect the bases AB and BE; and the triangle ADI being likewise isosceles, $IP = AP$, and consequently $IB = AE = 2AB$. But, from what has been formerly shown, it is evident that $AK^2 = 2AB^2$ and $AD^2 = 3AB^2$; and since $AE = 2AB$, $AE^2 = 4AB^2$. In the right angled triangles IBK and IBG, $IK^2 = IB^2 + BK^2 = 4EB^2 + BK^2 = 5AB^2$, $IG^2 = IB^2 + BG^2 = 4AB^2 + 2AB^2 = 6AB^2$; but (II. 26.) $IC^2 = IB^2 + BC^2 + IB.2BO = 4AB^2 + AB^2 + 2AB^2 = 7AB^2$. Again, GH being double of BG, $GH^2 = 4 \times 2AB^2 = 8AB^2$, and AI being the triple of AE, $AI^2 = 9AB^2$; and lastly, IAL being a right angled triangle, $IL^2 = IA^2 + AL^2 = 9AB^2 + AB^2 = 10AB^2$.

If AB, therefore, denote the unit of any scale, it will follow, that $AK = \sqrt{2}$, $AD = \sqrt{3}$, $IK = \sqrt{5}$, $IG = \sqrt{6}$, $IC = \sqrt{7}$, $GH = \sqrt{8}$, and $IL = \sqrt{10}$.

GEOMETRICAL ANALYSIS.

GEOMETRICAL ANALYSIS.

Analysis is that procedure by which a proposition is traced up, through a chain of necessary dependence, to some known operation, or some admitted principle. It is alike applicable to the investigation of truth contemplated in a *theorem*, or to the discovery of the construction required for a *problem*. Analysis, as its name indeed imports, is thus a sort of inverted form of solution. Assuming the hypothesis advanced, it remounts, step by step, till it has reached a source already explored. The reverse of this process constitutes *Synthesis*, or *Composition*,—which is the mode usually employed for explaining the elements of science. Analysis, therefore, presents the medium of invention; while synthesis naturally directs the course of instruction *.

* See Note LI.

BOOK I.

DEFINITIONS.

1. *Quantities* are said to be *given*, which are either exhibited, or may be found.

2. A *ratio* is said to be *given*, when it is the same as that of two given quantities.

3. *Points, lines,* and *spaces*, are said to be *given in position*, if they have always the same situation, and are either actually exhibited, or may be found.

4. A *circle* is *given in position*, when its centre is given; it is *given in magnitude*, if its radius be given.

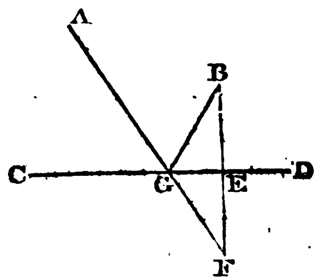
5. *Rectilineal figures* are said to be *given in species*, when figures similar to them are given.

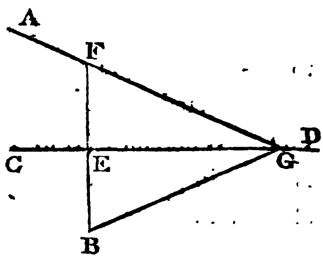
PROP. I. PROB.

From two given points, to draw straight lines, making equal angles at the same point in a straight line given in position.

Let A, B be two given points, and CD a straight line given in position; it is required to draw AG and GB, so that the angles AGC and BGD shall be equal.

ANALYSIS.





From B, one of the given points, let fall the perpendicular BE, and produce it to meet AG, or its extension in F. The angle BGE, being equal to AGC, is equal to the vertical angle FGE, the right angle BEG is equal to FEG, and the side GE is common to the triangles GBE and GFE, which (I. 21. El.) are therefore equal, and hence the side BE is equal to FE. But the perpendicular BE is given, and consequently FE is given both in position and magnitude; whence the point F is given, and therefore G the intersection of the straight line AF with CD.

COMPOSITION.

Let fall the perpendicular BE, and produce it equally on the opposite side, join AF meeting CD in G; AG and BG are the straight lines required.

For the triangles GBE and GFE, having the side BE equal to FE, GE common, and the contained angle BEG equal to FEG, are (I. 3. El.) equal; and consequently the angle BGE is equal to FGE or AGC.

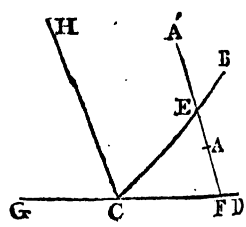
PROP. II. PROB.

Through a given point, to draw a straight line at equal angles with two straight lines given in position.

Let A be the given point, and CB, CD the straight lines which are given in position.

ANALYSIS.

Draw (I. 24. El.) CH parallel to FE, and produce DC. The exterior angle GCH (I. 32. El.) is equal to CFE, and ECH is equal to the alternate angle CEF; but the angle CFE is equal to CEF, and consequently GCH is equal to ECH, and the angle GCE is thus bisected by the straight line CH. Wherefore (I. 5. El.) CH is given in position, and hence (I. 24. El.) the parallel EF is also given.



COMPOSITION.

Bisect (I. 5. El.) the adjacent angle GCB by the straight line CH, and parallel to this draw EF (I. 24. El.) through the given point A; the angle CEF is equal to CFE. For these angles are equal to the exterior and alternate angles GCH and ECH, and are consequently equal to each other.

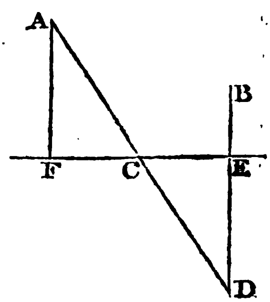
PROP. III. PROB.

Through a given point, to draw a straight line, such that the segments intercepted by perpendiculars let fall upon it from two given points, shall be equal.

The points A, B, and C being given,—to draw a straight line FE, so that the parts CF and CE, cut off by the perpendiculars AF and BE, shall be equal.

ANALYSIS.

Produce AC to meet the extension of BE in D. The right angled triangles AFC and DEC, having the vertical angle ACF equal to DCE, and the side CF equal to CE, are (I. 21. El.) equal, and hence the side CA is equal to CD. But CA is evidently given; wherefore CD and the point D are given; BD is consequently given, and hence the perpendicular CE is given.



COMPOSITION.

Produce AC till CD be equal to it, join BD, and draw CE perpendicular, and AF parallel to it: FCE is the line required. For the triangles FAC and EDC, having the angles ACF, AFC equal to DCE, DEC, and the side AC equal to CD,—are equal, and consequently CF is equal to CE.

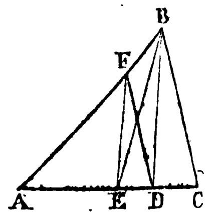
PROP. IV. PROB.

To bisect a given triangle, by a straight line drawn from a given point in one of its sides.

Let it be required, from the point D, to draw DF, bisecting the triangle ABC.

ANALYSIS.

Bisect (I. 7. El.) the side AC in E, and join EB, EF and BD. The triangle ABE is (II. 2. El.) equal to EBC, and is consequently the half of ABC; wherefore ABE is equal to AFD, and, taking AFE from both, the remaining triangle EFB is equal to EFD; and since these triangles stand on the same base, they must (II. 3. El.) have the same altitude, or EF is parallel to BD.



But the points B and D being given, the straight line BD is given in position, and consequently EF is also given in position.

COMPOSITION.

Having bisected AC in E and joined BD, draw EF parallel to it, meeting AB in F; the straight line DF divides the triangle ABC into two equal portions.

For join BE. Because BD is parallel to EF, the triangle EFB (II. 1. El.) is equal to EFD; and, adding AFE to each, the triangle AFD is equal to ABE, that is, to the half of the triangle ABC.

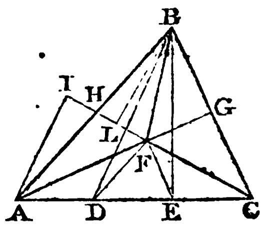
PROP. V. PROB.

To find a point within a given triangle, from which straight lines drawn to the several corners will divide the triangle into three equal portions.

Let F be the required point, from which the lines FA, FB, and FC trisect the triangle ABC.

ANALYSIS.

Draw FD, FE parallel to the sides BA, BC, and join BD, BE. Since FD is parallel to AB, the triangle ABF (II. 2.) El.) is equal to ABD, which is hence the third part of ABC; and, for the same reason, the triangle BFC is equal to BEC, which is also the third part of ABC. Wherefore the bases AD and EC are each the third part of AC, and consequently the points of section D and E are given; hence (I. 24.) El.) the parallels DF and EF are given in position, and their point of concourse is therefore given.



But the point F may be determined otherwise. For produce AF and CF to G and H. The triangle DFE is evidently (I. 31. El.) similar to ABC, and therefore AC : AB :: DE : DF, but AC=3DE, and consequently (V. 8. and 5. El.) AB=3DF. Again, because AH and DF are parallel AC : AH :: DC : DF, and (V. 13. El.) 2AC : 2AH :: 3DC : 3DF; but 2AC=6AD=3DC, and 2AH=3DF=AB. Hence AB is bisected in H; and, for the same reason, BC is bisected in G. Wherefore the points H and G being thus given, the intersection F of the straight lines CH and AG is likewise given.

COMPOSITION.

Bisect AB and BC (I. 7. El.) in H and G, join CH and AG, and, from their point of intersection, draw FA, FB, and FC; the triangle ABC will thus be divided into three equal portions.

For, from the points A and B let fall the perpendiculars AI and BL. The triangles HAI and HBL, having the angles AHI and AIH equal to BHL and BLH, and the side AH equal to BH, are (I. 21. El.) equal, and consequently AI= BL. The triangles AFC and BFC, standing on the same base CF, and having equal altitudes AI and BL, are equal (II. 2. El.). And, in the same manner, it is shown that the triangles AFC and AFB are equal. Wherefore the whole triangle ABC is divided into three equal triangles, having their common vertex at the point F.

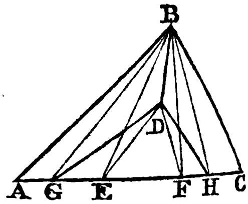
PROP. VI. PROB.

To trisect a given triangle, by straight lines drawn from a given point within it.

Let ABC be a triangle which it is required to divide into three equal portions, by the straight lines DB, DG, and DH, drawn from the point D.

ANALYSIS.

Join BG, draw DE (I. 24. El.) parallel to it, and join BE. The triangle BDG is equal to BEG, and consequently the compound space ABDG is equal to the triangle ABE, which is, therefore, the third part of the triangle ABC. Hence the base AE is the third part of AC, and the point E is consequently given; wherefore the parallel BG is given, and also the point G and DG. In like manner, joining BH, drawing DF parallel to it,—and joining DH, it may be shown that BH is given.



COMPOSITION.

Trisect (I. 38. El.) the base AC in the points E and F, join DE, DF, and parallel to these draw BG, BH, and join

DB, DG, DH; the triangle ABC is thus divided into three equal portions.

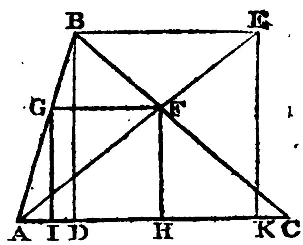
For DE being parallel to BG, the triangle BDG is equal to BEG, and therefore the space ABDG is equal to the triangle ABE. In the same manner, it is shown that the space BDHC is equal to the triangle BFC; and consequently the remaining triangles GDH and EBF are equal. But the triangles ABE, EBF, and FBC, standing on equal bases, are equal; wherefore the spaces ABDG, GDH, and BDHC, are each of them the third part of the original triangle ABC.

PROP. VII. PROB.

To inscribe a square in a given triangle.

Let ABC be the triangle in which it is required to inscribe a square IGFH.

ANALYSIS.



Join AF, and produce it to meet a parallel to AC in E, and let fall the perpendiculars BD and EK.

Because EB is parallel to FG or AC, AF : AE : : FG : EB (VI. 2. El.); and since the perpendicular EK is parallel to FH, AF : AE : : FH : EK. Wherefore FG : EB : : FH : EK; but FG = FH, and consequently (V. 8. and 5. El.) EB = EK. Again, EK, being equal to BD, the altitude of the triangle ABC is given, and, therefore, EB is given both in position and magnitude; whence the point E is given, and the intersection of AE with BC is given, and consequently the parallel FG and the perpendicular FH are given, and thence the square IGFH.

COMPOSITION.

From B draw BD perpendicular and BE parallel, to AC, make BE equal to BD, join AE, intersecting BC in F, and complete the rectangle IGFH.

Because BE and EK are parallel to GF and FH, AE : AF : : BE : GF, and AE : AF : : EK : FH; wherefore BE : GF : : EK : FH; but BE=EK, and consequently GF=FH. It is hence evident that IGFH is a square.

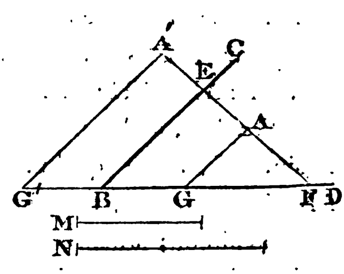
PROP. VIII. PROB.

To draw a straight line through a given point, so that its portions, terminated by two straight lines given in position, shall have a given ratio.

Let A be a given point, and BC, BD two straight lines given in position; it is required to draw EAF, such that EA shall be to AF as M to N.

ANALYSIS.

Draw AG parallel to BC, and meeting BD in the point G, which is thus given. The diverging lines FE, FB are cut proportionally by parallels BE, GA, (VI. 1. El.), and consequently EA : AF : : BG : GF; but the ratio of EA to AF is given, and therefore the ratio of BG to GF; and BG being given, GF is given, and the point F, and hence the straight line EAF is given.



COMPOSITION.

Draw AG parallel to BC, make (VI. 3. El.) BG : GF : : M : N, and join FAE.

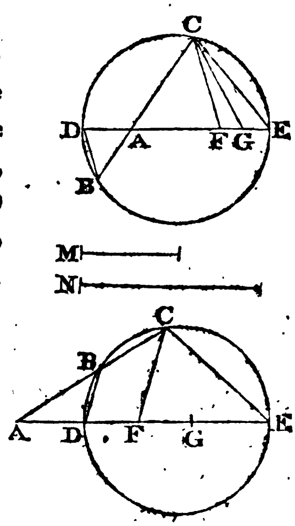
For, BE and AG being parallel, EA : AF : : BG : GF; but BG : GF : : M : N, and therefore EA : AF : : M : N.

PROP. IX. PROB.

Through a given point, to draw a straight line that shall be cut in a given ratio, by the circumference of a given circle.

Let A be the given point, and BDCE the given circle; it is required to draw BC, so that BA shall be to AC as M to N.

ANALYSIS.



Draw the diameter DAE, join DB, CE, and draw CF parallel to DB. Because the point A and the centre of the circle are given, the diameter DE is given in position, and consequently its extremities D and E. But, DB being parallel to CF, BA : AC : : DA : AF (VI. 1. El.), wherefore the ratio of DA to AF is given, and since DA is given, AF is also given. Again, BA.AC =AD.AE (III. 32. El.), and consequently AE : AC : : BA : DA; but BA : DA : : AC : AF (VI. 1. El.), whence AE : AC : : AC : AF, or AC is a mean proportional between AF and AE, and is, therefore, given. The point C is thus given, and consequently BC.

COMPOSITION.

Having drawn the diameter DE, make DA : AF : : M : N, and (VI. 13. El.) AG a mean proportional between AF and

AE, and inflect AC equal to it; BAC is the straight line required.

For join DB, CF, and CE. Since the rectangle BA, AC is equal to the rectangle DA, AE, it follows that AE : AC :: BA : DA; but, by construction, AE : AC :: AC : AF, and therefore AC : AF :: BA : DA; hence (VI. 1. cor. 1. El.) CF is parallel to DB, and consequently BA is to AC, as DA to AF, that is, as M to N.

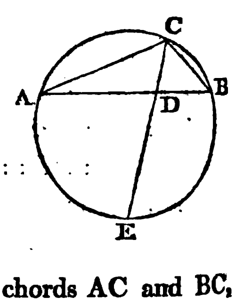
PROP. X. PROB.

'From two given points in the circumference of a given circle, to inflect, to another point in the circumference, straight lines that shall have a given ratio.

From the points A and B, let it be required to inflect AC and BC in a given ratio.

ANALYSIS.

Draw (I. 5. El.) CE bisecting the vertical angle ACB. Therefore (VI. 11. El.) AC : CB :: AD : DB, and consequently the ratio of AD to DB is given, and thence (VI. 4. El.) the point D is given. But since the angle ACE is equal to BCE, the arc AE is (III. 18. cor. El.) equal to the arc EB, and therefore the point E is given. Whence the points E and D being given, the straight line EDC is given in position, and consequently the point C and the chords AC and BC, are given.



COMPOSITION.

Bisect (III. 15. El.) the arc AEB in E, divide AB (VI. 4. El.) in the given ratio at D, join ED, and produce it to meet

the opposite circumference in C; the chords AC and CB are in the given ratio.

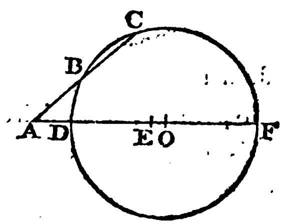
For since the arc AE is equal to BE, the angle ACD is (III. 18. cor. El.) equal to BCD, and consequently (VI. 11. El.) AC : CB : : AD : DB, that is, in the given ratio.

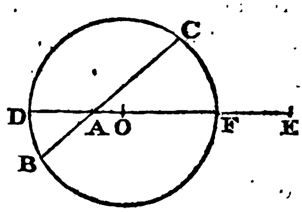
PROP. XI. PROB.

Through a given point, to draw a straight line to a circle, so that the rectangle under the part limited by the circumference and the segment included within the circle, shall be equal to a given space.

Let it be required through the point A to draw ABC, such that the rectangle AB, BC shall be equal to a given space.

ANALYSIS.





Through the centre O draw AF, and (II. 9. El.) find AE, which forms with AD a rectangle equal to the given space. Because (III. 32. El.) AB.AC= AD.AF, and, by construction, AB.BC=AD.AE; it follows (V. 6. El.) that AD : AB : : AC : AF : : BC : AE; whence (V. 19. cor. 1. El.) AD : AB : : AC —BC or BC—AC, that is AB : AF—AE or AE—AF, that is EF. Wherefore AB is a mean proportional between AD and EF; but AE being given, EF is also given, and consequently AB is given both in magnitude and position.

COMPOSITION.

Draw AF through the centre of the circle, make (II. 9. El.) the rectangle AD, AE equal to the given space, find (VI. 18.

El.) a mean proportional to AD and EF, and inflect this from A towards B; the rectangle AB, BC is equal to the given space.

For (V. 6.) AD : AB : : AB : EF, and (V. 6. and III. 32. El.) AD : AB : : AC : AF, whence (V. 19. cor. 1. El.) AD : AB : : AC$\mp$AB or BC : AF$\mp$EF or AE, and consequently AD.AE=AB.BC.

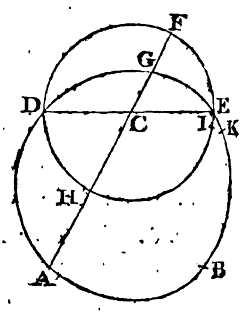
PROP. XII. PROB.

Through two given points, to describe a circle bisecting the circumference of a given circle.

Let A and B be two points, through which it is required to describe a circle ADGEB, that shall bisect the circumference of the circle HDFE.

ANALYSIS.

Join D, E, the points of intersection. Because DFE is, by hypothesis, a semicircumference, DE is a diameter, and must, therefore, pass through the centre C. Join AC, and produce it to F. Since DC =CE, it is evident (III. 32. El.) that AC.CG=DC^2=HC.CF; but the rectangle HC, CF is given, and consequently the rectangle AC, CG is also given; and AC being given, CG is hence given, and the point G.



Wherefore the three points A, G, and B being given, the circle AGB is (III. 10. El.) given.

COMPOSITION.

Through C, the centre of the given circle, draw ACF, make (VI. 3. El.) AC : HC : : CF or HC : CG, and through

the three points A, G, and B, describe (III. 10. cor. El.) the circle AGB: This will bisect the circumference HDFE.

For, through one of the points of intersection, draw the diameter DCI, and produce it to meet the circumference of the circle AGB in K. Because AC : HC : : HC : CG, the square of HC is (V. 6. El.) equal to the rectangle AC and CG; but (III. 32. El.) $HC^2 = DC.CI$, and $AC.CG = DC.CK$; wherefore $DC.CI = DC.CK$, and (II. 3. cor. El.) $CI = CK$, or the points I and K are one, and the circle AGB passes through both extremities of the diameter of HDFE.

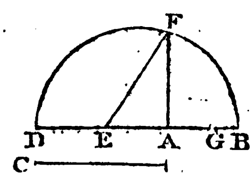
PROP. XIII. PROB.

To cut a given straight line, such that the square of one part shall be equivalent to the rectangle under the remainder and another given straight line.

Let AB be a straight line, from which it is required to cut off a segment whose square shall be equivalent to the rectangle under the remainder and the straight line C.

ANALYSIS.

Produce BA till AD be equal to C, on DB describe a semicircle and erect the perpendicular AF. Because $AG^2 = C \times GB$, it follows (V. 6. El.) that DA : AG : : AG : GB; wherefore (V. 19. El.) DA : AG : : DG : AB, and consequently $DA.AB = AG.DG$; but (III. 32. cor. 1. El.) $DA.AB = AF^2$, and therefore $AG.DG = AF^2$; whence AF is equal to a tangent



drawn from G to a semicircle described on DA. Bisect DA in E, and join EF; and because $AG.DG = AF^2$, add EA^2

to each, and AG.DG + EA^2, or (II. 19. cor. 2. El.) EG^2, is equivalent to $AF^2 + EA^2$ or (II. 11.) EF^2; whence EG is equal to EF, and is therefore given.

COMPOSITION.

Having produced AD equal to C, and described on BD a semicircle, erect the perpendicular AF, bisect AD in E, join EF and make EG equal to it; the square of the segment AG thus formed in AB is equivalent to the rectangle under the remaining part GB and the given line C.

For EFA being a right-angled triangle $EF^2 = EA^2 + AF^2$ (II. 11. El.), and consequently $AF^2 = EF^2 - EA^2$, or $EG^2 - EA^2$; and since (II. 19. El.) $EG^2 - EA^2 = (EG + EA).(EG - EA)$, or DG.AG, therefore $AF^2 = DG.AG$. But (III. 32. cor. 1. El.) $AF^2 = DA.AB$; whence DG.AG = DA.AB, and AG : AB : : DA : DG (VI. 6. El.); wherefore (V. 11. and V. 7. El.) AB—AG, or GB : AG : : DG—DA, or AG : DA, whence (V. 6. El.) $AG^2 = GB.DA$.

Cor. If DA, or C, be equal to AB, then $AG^2 = AB.BG$, or AB : AG : : AG : BG, and, therefore, the line AB is now divided in extreme and mean ratio, at the point G. The construction also becomes evidently the same with that which was given in Book II. Prop. 22. of the Elements, for the medial section of a line, and which is really a simple case of the same problem.

PROP. XIV. PROB.

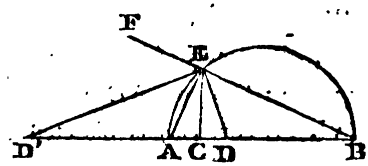
To divide a straight line, such that its segments shall have the subduplicate ratio of those formed by another section of the same kind.

Let it be required to divide the straight line AB in D, such that the segments AD, DB shall be in the subduplicate ratio of other like segments AC, CB.

1. *Let the given section be internal.*

ANALYSIS.

On AB describe a semicircle, erect the perpendicular CE, and join AE, BE and ED or ED′. Because (III. 22. El.) AEB is a right angle, the ratio of AE to BE (VI. 16. cor. 1. El.) is the subduplicate of that of AC to BC, and consequently AE : BE :: AD : BD, or AD′ : BD′; wherefore (VI. 11. cor. El.) the vertical angle AEB is bisected internally or externally by ED or ED′. But the perpendicular and the semicircle being both given,—the vertex E, the straight line ED or ED′, and the point of section D or D′, are likewise given.



COMPOSITION.

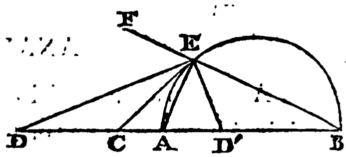
Having on AB described a semicircle, erect the perpendicular CE, join EA, EB, and draw ED or ED′ bisecting the angle AEB or its adjacent angle AEF; the internal segments AD, DB, or the external segments AD′, D′B, are in the subduplicate ratio of AC to CB.

For (VI. 11. El.) AE : BE :: AD : DB or AD′ : D′B; but the triangle AEB being right-angled, AE is to BE (VI. 16. cor. El.) in the subduplicate ratio of AC to CB, and consequently AD is to BD or AD′ to D′B in the same subduplicate ratio.

2. *Let the given section be external.*

ANALYSIS.

On AB describe a semicircle, draw the tangent CE, and join AE, BE and ED or ED′. The triangles ACE and ECB are similar, for (III. 25. El.) the angle CEA is equal to CBE in the alternate segment, and BCE is common to both trian-



gles; whence AC : CE : : CE : BC, and consequently (V. def. 20. El.) the ratio of CE to BC is the subduplicate of that of AC to BC. But in these similar triangles, AE : CE : : BE : BC, and alternately AE : BE : : CE : BC; wherefore AE : BE : : AD : DB, or AD′ : D′B, and the vertical angle AEB (VI. 11. cor. El.) is bisected externally or internally by ED or ED′.

COMPOSITION.

Having described a semicircle on AB, apply (III. 26. El.) the tangent CE, join AE, BE, and draw ED or ED′ bisecting externally or internally the vertical angle AEB; the external segments AD, DB, or the internal segments AD′, D′B are in the subduplicate ratio of AC to BC.

For the angle CEA being (III. 25. El.) equal to CBE, and BCE common to the two triangles ACE and ECB, these are similar, and AC : CE : : CE : BC; whence the ratio of CE to BC is the subduplicate of that of AC to BC. Again, from the same similar triangles, AE : CE : : BE : BC, or alternately AE : BE : : CE : BC, and therefore AE is to BE in the subduplicate ratio of AC to BC. But (VI. 16. El.) AE : BE : : AD : DB, or AD′ : D′B, and consequently the ratio of AD to DB or of AD′ to D′B is the subduplicate of that of AC to BC.

Cor. In the second case, the angle CD′E (I. 32. El.) being equal to D′EB and D′BE, which are equal to D′EA and AEC, is therefore equal to CED′, and the triangle D′CE is hence isosceles. Again the angle DEF, equal by hypothesis to DEA or CED and AEC, is (I. 32. El.) equal to CDE and DBE or AEC, and consequently the triangle DCE is likewise isosceles. Wherefore CE = CD = CD′, and thus, without bisecting the vertical angle, the point D or D′ is found from the

tangent CE, which is a mean proportional between the segments AC and BC.

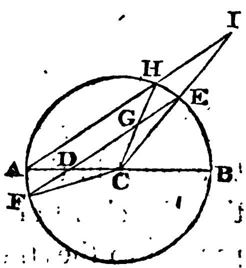
PROP. XV. PROB.

To find a point in the diameter of a circle, such that the square of a straight line inflected from it at a given angle to the circumference, shall have a given ratio to the rectangle under the segments of the diameter.

Let it be required to draw DE at a given angle with DB, and so that the square of DE shall have a given ratio to the rectangle AD, DB.

ANALYSIS.

Make EG = FD, join CF, draw the radius CGH, join AH, and produce it to meet the extension of CE in I.



Because CE is equal to CF, the angle CEF is (I. 11. El.) equal to CFE. Wherefore the triangles CGE and CDF, having thus the angle CEG equal to CFD, and the sides CE and EG equal to CF and FD, are (I. 3. El.) equal, and consequently the angle ECG is equal to FCD; whence (III. 13. El.) the arc HE is equal to AF, and therefore (III. 20. cor. El.) AH is parallel to DE. But the angle BDE is given, and thence BAH; wherefore the chord AH is given. Again, the rectangle AD,DB, being equal to FD,DE (III. 32. El.), is also equal to DE,EG; and therefore DE^2 is to DE.EG, or (V. 25. cor. 2. El.) DE is to EG, in the given ratio; but (VI. 2. El.), DE : EG :: AI : IH, consequently AI is to IH in a given ratio, and hence AH is to HI in a given ratio. Wherefore

since AH is given, IH and the point I are given; and thence IC, the point E, and DE, are all given.

COMPOSITION.

Draw AH at an inclination with AB equal to the given angle, and produce it to I, so that AI shall be to IH in the given ratio, join IC, and draw ED parallel to IA; D is the point required.

Because AI : IH : : DE : EG, DE is to EG in the given ratio, and consequently DE^2 is to DE.EG in the same ratio. But FE being parallel to AH, the arc HE is equal to AF, and thence the angle HCE is equal to ACF; the triangles CGE and CDF, having thus the side CE equal to CF, and the angles ECG and CEG equal to FCD and CFD,—are (I. 21. El.) equal, and hence the side EG is equal to FD. Wherefore DE.EG=DE.FD=AD.DB, and consequently DE^2 is to AD.DB in the given ratio.

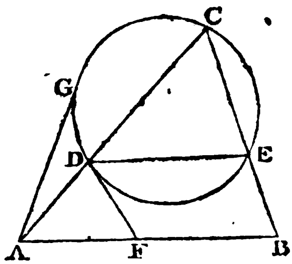
PROP. XVI. PROB.

Through two given points, to draw straight lines to a point in the circumference of a given circle, so that the chord of the intercepted segment shall be parallel to the straight line which connects the given points.

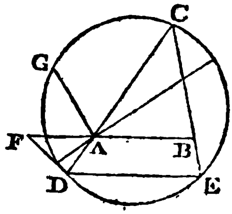
Let it be required, from the points A and B, to inflect AC and BC cutting the given circumference in D and E, such that DE shall be parallel to AB.

ANALYSIS.

Draw the tangent DF meeting AB in F. The angle FDE is equal to the angle ECD or its supplement in the alternate segment (III. 25. El.); but DE being parallel to AB, the



angle FDE or its supplement is (I, 23. El.) equal to the alternate angle AFD, which is consequently equal to the angle ECD or ACB; wherefore the triangles ADF and ABC, having likewise a common angle CAB, are similar, and AD : AF : : AB : AC, and hence AD.AC = AF.AB. But since the point A and the circle DCE are given, the rectangle AD, AC is also given; for it is equal to the square of the



tangent AG (III. 32. cor. 2. El.), when A lies without the circumference,—and equal to the square of AG (III. 32. cor. 1. El.) a perpendicular to the diameter, in the case where that point lies within the circle. Hence the rectangle AF, AB is given; and AB being given, AF is likewise given, and consequently the point F. Wherefore the tangent FD is given in position; and since the point A is given, the straight line AC is given, and thence BC and the intersection E.

COMPOSITION.

If the point A be without the circle, draw the tangent AG; or if it lie within the circle, erect AG perpendicular to the diameter which passes through it. Make (VI. 3. El.) AB : AG : : AG : AF, from F draw the tangent FD, join AD, and produce it to meet the opposite circumference in C, join CB, cutting the circle in E; the straight line DE is parallel to AB.

For, since AB : AG : : AG : AF, $AG^2 = AB.AF$; but (III. 32. cor. 1. and 2. El.) $AG^2 = CA.AD$, whence AB.AF = CA.AD, and consequently (V. 6 El.) AB : AC : : AD : AF. Wherefore (VI. 14. El.) the triangles BAC and DAF, having the sides about their common angle proportional, are similar,

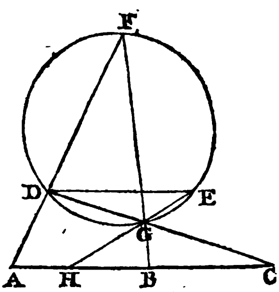
and hence the angle ACB is equal to AFD; but (III. 25. El.) ACB or DCE is equal to EDF or its supplement, and consequently the angle AFD is equal to EDF or its supplement, and (I. 23. cor. El.) the chord DE is parallel to AB.

PROP. XVII. PROB.

From two given points, to inflect straight lines to the circumference of a circle, such that the chord of their intercepted arc shall tend to a given point in the direction of the former.

Let it be required, from the points A and B, to inflect AF and BF, so that the chord DG produced shall meet the extension of AB in the point C.

ANALYSIS.



Draw DE parallel to AC, join EG, and produce it to meet AB in H.

The angle BHG is equal to the alternate angle GED, which is equal (III. 18. El.) to GFD, and consequently the angles BHG and BFA are equal, and the triangles BGH and BAF are similar. Wherefore BG : BH : : BA : BF, and BG.BF = BH.BA; but the rectangle BG, BF is given, since it is equal to the square of a tangent drawn from B, and hence BH.BA is given, and the point H. The problem is thus reduced to the last Proposition, and only requires, from the points C and H, to inflect CD and HE, such that DE, the chord of their intercepted arc, may be parallel to HC.

COMPOSITION.

From the point B draw a tangent BI to the circle, make BA : BI : : BI : BH, and, by the last Proposition, inflect HE and CD such that DE shall be parallel to HC; then BG, being produced to F in the circumference, ADF forms one straight line.

For since BA : BI : : BI : BH, the rectangle BA, BH will be equivalent to the square of BI or (III. 32. cor. 2. El.) to the rectangle BG, BF; consequently (V. 6.) BA : BF : : BG : BH, and (VI. 14. El.) the triangles BAF and BGH are similar; wherefore the angle BFA is equal to BHG which (I. 23. El.) is equal to GED, and this again (III. 18. El.) is equal to GFD; whence BFA is equal to GFD, or the straight lines FA and FD lie in the same direction from F.

PROP. XVIII. PROB.

From two given points in the circumference of a given circle, to inflect straight lines to another point in the opposite circumference, such as to intercept, on either side of the centre, equal segments of a given diameter.

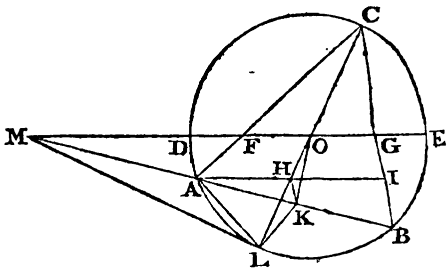
Let it be required, from the points A and B, to inflect AC and BC, so as to intercept, on the diameter DE, equal portions from the centre.

ANALYSIS.

Join BA, and produce it and the diameter ED to meet in M, draw COL, from O let fall the perpendicular OK upon AB, join LK, through A draw AHI parallel to DE, and join HK.

The parallels FG and AI are cut proportionally by the diverging lines CA, CH, and CI (VI. 1. El.); but FO is equal

to OG, and consequently AH is equal to HI. Wherefore (II. 4. El.) HK is parallel to IB, and the angle AKH is equal to ABI (I. 23. El.); and since the angle ABI or ABC is equal to ALC (III. 18. El.), the angle AKH is equal to ALC or ALH, and hence (III. 18. cor. El.) the quadrilateral figure AHKL is contained in a circle. Consequently (III. 18. El.) the angle HAK is equal to HLK; but HAK is equal (I. 23. El.) to OMK, which is therefore equal to HLK or OLK, and thence the quadrilateral figure MOKL is also contained in a circle.



Wherefore (III. 18. El.) the angle MLO is equal to MKO; but MKO is a right angle, and consequently MLO is likewise a right angle, and thence (III. 24. El.) ML is a tangent. But the point M, being the concourse of ED and BA, is given, and therefore the tangent ML to the given circle is given (III. 26. El.); whence the diameter LC, and the point C, are given.

COMPOSITION.

Produce ED and BA to meet in M, draw the tangent ML and the diameter LC; the straight lines AC and BC will cut off from the centre equal portions, OF and OG, of the given diameter ED.

For draw AI parallel to DE, and OK perpendicular to AB, and join LK and KH.

Because ML is a tangent, MLO is a right angle, and, therefore, equal to MKO; consequently (III. 18. El.) MKL is equal to MOL, that is, (I. 23. El.) to AHL. Wherefore the quadrilateral figure AHKL is contained in a circle, and

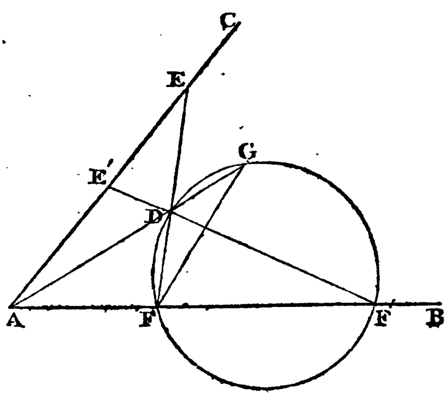
hence (III. 18. El.) the angle ALH is equal to AKH; but, for the same reason, ALH or ALC is equal to ABC or ABI, and consequently AKH is equal to ABI, and (I. 23. El.) KH parallel to BI. Now since AK is equal to KB, it follows that AH is equal to HI, and hence that FO is equal to OG.

PROP. XIX. PROB.

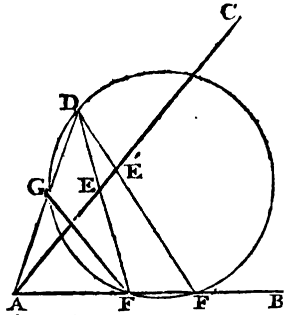
Through a given point to draw a straight line, so that the rectangle under its segments, intercepted by two straight lines given in position, shall be equal to a given space.

Let AB, AC be two straight lines, and D a point through which it is required to draw EF, such that the rectangle under its segments ED, DF shall be equal to a given space.

ANALYSIS.



Join AD, from F draw (I. 4. El.) FG, making an angle DFG equal to DAE, and meeting AD or its production in G. The triangles ADE and FDG, being thus evidently similar, AD : ED : : DF : DG,



and consequently (V. 6. El.) AD.DG=ED.DF. But the rectangle ED, DF is given, and therefore also the rectangle AD, DG; and since AD is given in position and magnitude, DG and the point G are given. Again, the angle DFG, being equal to DAC, is given, and thence (III. 27. El.) the segment of the circle which contains it; wherefore the contact or intersection of that arc with the straight line AB is given, and consequently the position of EF or E′F′ is likewise given.

COMPOSITION.

Join AD, make the rectangle AD, DG equal to the given space, and on DG describe (III. 27. El.) an arc containing an angle equal to DAC, and meeting AB in F or F′; EDF or EDF′ is the straight line required.

For the triangles ADE and FDG are similar, and consequently (VI. 12. El.) AD : ED : : DF : DG; whence (V. 6. El.) ED.DF=AD.DG; but the rectangle AD.DG is equal to the given space, and therefore the rectangle ED.DF is also equal to that space.

A limitation evidently takes place, when the points F and F′ coincide, and the circle touches the straight line AB. In this case, the angle AFD or BFD, being equal to DGF in the alternate segment, is therefore equal to AED, and consequently AFE is equal to AEF, and (I. 12. El.) AF=AE.

PROP. XX. PROB.

Two straight lines being given, to draw, through a given point, another straight line, cutting off seg-

ments which are together equal to a given straight line.

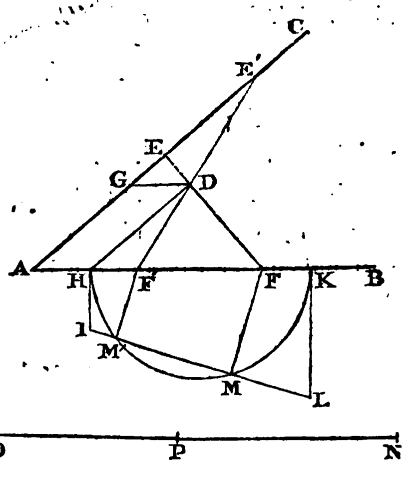
Let AB, AC be two straight lines, and D a given point, through which it is required to draw a straight line EF, so as to cut off the segments AE and AF, that are together equal to ON.

The point D may lie either within or without the angle formed by the straight lines AB and AC.

1. *Let D have an internal position.*

ANALYSIS.

Draw DG and DH (I. 24. El.) parallel to AB and AC. Because the point D is given, and AB,AC are given in position, the parallelogram AGDH is given. And since the triangles EDG and DFH are evidently similar, EG : GD : : DH : HF, and therefore EG.HF = GD.DH. But AG and AH, or DH and GD, being given, the rectangle GD, DH is given, and therefore EG,HF is given. Make FK=EG, and the rectangle HF, FK is hence given; but HK, being equal to HF and FK or the excess of AF and AE above GD and DH, is given, and consequently (VI. 19. El.) its segments HF, FK are given; whence the point H being given, the point of section F or F′, and the straight line EDF or E′DF′, are given.



COMPOSITION.

Draw the parallels DG and DH. From ON, the sum of the two segments AE and AF, cut off OP=AG+AH, and

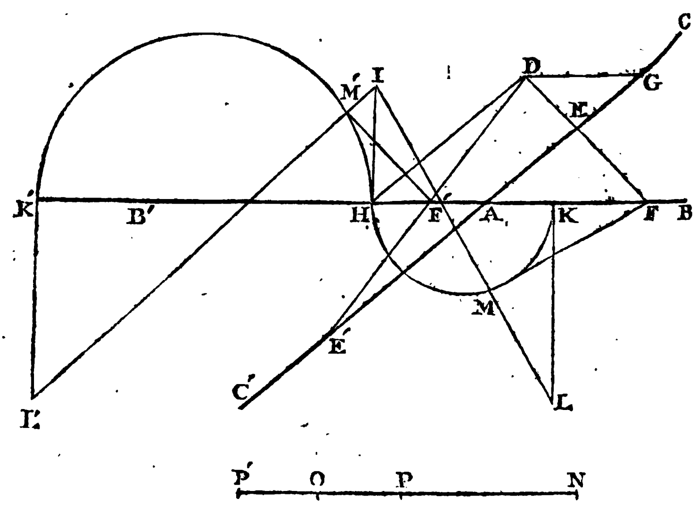
make HK=PN. On HK describe a semicircle, from the extremities of the diameter erect the perpendiculars HI and KL equal to AH and AG, join IL, and at right angles to this, and from the point or points where it meets the circumference, draw MF or M′F′; EDF or E′DF′ is the straight line required.

For (VI. 19. El.) HI.KL=HF.FK, and consequently AH.AG=HF.FK. But, from the similar triangles EGD and DHF, EG : GD, or AH : : DH, or AG : HF, and therefore (V. 6. El.) AH.AG=HF.EG; whence HF.FK= HF.EG, and FK=EG. And since AG+AH=OP, and HF+EG=HK=PN, it follows that AG+EG+AH+ HF, or AE+AF=ON.

2. *Let the point D have an external position with respect to the straight lines AB and AC.*

ANALYSIS.

Draw DG parallel to AB, and DH parallel to AC and meeting AB produced. The triangles EDG and DHF being similar, EG : DG : : DH : HF, and (V. 6. El.) EG.HF= DG.DH; but DG and DH are both given, and hence the rectangle under EG and HF is given. Make FK=EG, and



therefore $HK = HF - EG = DG + AF - (DH - AE) = AF + AE - (DH - DG)$; whence HK and the rectangle HF,FK are given, and consequently (VI. 19. El.) the point F is given.

If DF′E′ intersect the straight lines AB and AC on the other side of their vertex A, the triangles E′DG and DF′H are still similar, and $E'G : DG :: DH : HF'$; wherefore $E'G.HF'$, being equal to $DG.DH$, is given. Make $F'K' = E'G$, and thence $HK' = E'G - HF' = AE' + DH - (DG - AF') = AF' + AE' + (DH - DG)$; consequently HK′ and the rectangle $HF'.F'K'$ are given, and therefore (VI. 19. El.) the point F is given.

COMPOSITION.

Make OP or OP′ equal to the difference of the parallels DH and DG, from H place likewise towards opposite parts $HK = PN$ and $HK' = P'N$, on HK and HK′ describe semicircles, from H erect the perpendicular HI equal to DG, and, from K and K′, the perpendiculars KL and K′L′, each equal to DH, join IL and IL′, and, at right angles to these, from the points of section M and M′, draw MF and M′F′; the straight lines DEF and DF′E′ will cut off segments from AB and AC, which are together equal to ON.

For (VI. 19. El.) $HF.FK = HI.KL = DG.DH$; but $DG.DH = HF.EG$, and consequently $HF.EG = HF.FK$, or $EG = FK$. Wherefore $HK = HF - EG = AF + AE - (DH - DG)$; and since $HK = PN = ON - (DH - DG)$, it follows that $AF + AE = ON$.

In like manner, it is shown that $E'G = F'K'$, and hence $HK' = E'G - HF' = AF' + AE' + (DH - DG)$; but $HK' = P'N' = ON + (DH - DG)$, and consequently $AF' + AE' = ON$.

PROP. XXI. PROB.

From one of the corners of a given square, to draw a straight line, such that its portion, intercept-

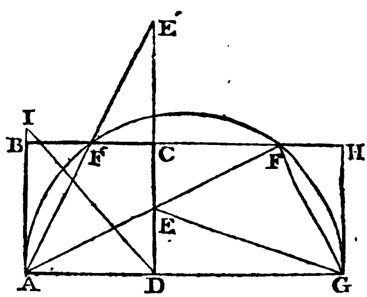
ed between the opposite sides of the figure, shall be equal to a given straight line.

Let ABCD be a square, and from the point A let it be required to draw AEF, so that the part EF, intercepted between CD and BC, or their extension, may be equal to a given straight line.

ANALYSIS.

Draw FG perpendicular to AF, meeting AD produced in G, from G let fall the perpendicular GH upon BC produced, and join EG.

The angle EFH is (I. 32. El.) equal to ECF and FEC, and it is also equal to EFG and GFH; consequently, ECF and EFG being right angles, the remaining angles FEC and GFH are equal; whence the triangles EAD and FGH, having the angle AED or CEF equal to GFH, the angles at D and H both right angles, and the side AD equal to GH or CD,—



are (I. 21. El.) equal, and therefore the side AE is equal to FG. But EFG and EDG being right-angled triangles, $EF^2+FG^2=EG^2=ED^2+DG^2$, (II. 11. El.), or $EF^2+AE^2=ED^2+DG^2$; but $AE^2=AD^2+ED^2$, and hence $EF^2+AD^2+ED^2=ED^2+DG^2$, or $EF^2+AD^2=DG^2$. Wherefore, since EF and AD are both given, DG is also given, and consequently AG; but the right angle AFG being contained in a semicircle described upon AG, the point F or F′, its contact or intersection with BC, is given, and consequently the straight line AEF.

COMPOSITION.

Make AI equal to the given straight line, join DI, and, equal to this, produce AD to G, upon AG describe a semicircle meeting the extension of BC in F or F′, and join AEF or AF′E′; EF, the external part of that straight line, is equal to AI.

For join FG, EG, and let fall the perpendicular GH upon BF. It is evident that $EF^2+FG^2=ED^2+DG^2$; and FG being equal to AE, $EF^2+AE^2=ED^2+DG^2$. But $AE^2=AD^2+ED^2$, and $DG^2=DI^2=AD^2+AI^2$; whence $EF^2+AD^2+ED^2=ED^2+AD^2+AI^2$, and therefore $EF^2=AI^2$, and EF=AI *.

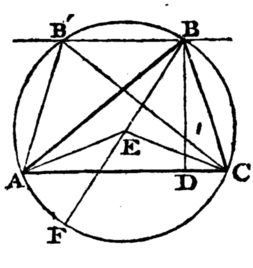
PROP. XXII. PROB.

Given the base of a triangle, its altitude, and the rectangle under its two sides,—to determine the triangle.

ANALYSIS.

About the triangle ABC describe (III. 10. cor. El.) a circle, and draw the diameter BF and the radii AE and CE.

Because the given rectangle AB.BC is (VI. 20. El.) equal to BD.BF, this rectangle is likewise given; and since the perpendicular BD is given, the diameter BF, and therefore the radii AE, CE, are given. But the base AC being given, the triangle AEC is hence given, and consequently the centre E and the circle ABCF are given. Again, because BD, the distance of the vertex of the triangle from its base, is given, that point must occur in the parallel



* See Note LII.

BB′, and, being thus placed in the contact or intersection of a given straight line with a given circle, is itself given.

COMPOSITION.

On BD construct (II. 9. El.) a rectangle equal to the given space, also form on AC the triangle AEC, having AE and CE each equal to half the greater side of that rectangle, from E with the radius EA describe a circle, on AC erect a perpendicular DB equal to the altitude of the triangle, and through B draw a parallel meeting the circumference in B or B′; *ABC* is the triangle required.

For ABC has evidently the given altitude BD, and the rectangle AB.BC, being equal (VI. 20. El.) to BF.BD, is therefore equal to the given space.

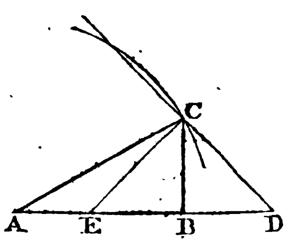
PROP. XXIII. PROB.

Given the hypotenuse of a right-angled triangle, and the sum or difference of the base and perpendicular, to construct the triangle.

ANALYSIS.

In the base AB, or its production, make BD or BE equal to the perpendicular BC, and join CD or CE.

The triangles CBD and CBE are right-angled and isosceles, and therefore the angles at D and E are each of them half a right angle. If AD, the sum of AB and BC, be given, the point D is given, and consequently the straight line DC, making a given angle with DA, is given in position; or if AE, the difference between the base and perpendicular, be given, the point E is given, and the straight line EC is given in position. But the hypotenuse AC being given, the point C must, therefore, occur in



the contact or intersection of a circle described from A with that radius and the straight line CD or CE. Consequently C is given, the perpendicular CB, and thence the right-angled triangle ABC.

COMPOSITION.

Make AD or AE equal to the sum or difference of AB and BC, draw (I. 5. and 4. El.) DC or EC at an angle CDE or CED equal to half a right angle, from A with the radius AC describe a circle meeting DC or EC in the point C, and from C (I. 6. El.) let fall the perpendicular CB : ACB is the triangle required.

For the right-angled triangles CBD and CBE are evidently isosceles, and therefore AD is equal to the sum, and AE to the difference, of AB and BC.

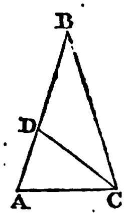
PROP. XXIV. PROB.

To investigate the construction of a regular pentagon or decagon.

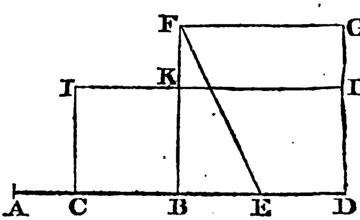
1. Every regular polygon is capable of being inscribed in a circle, and therefore the angles, formed at the centre by drawing radii to the several corners of the figure, are each of them equal to that part of four right angles corresponding to the number of sides. Consequently the central angles of a pentagon are each equal to the fifth, and those of a decagon are each equal to the tenth, part of four right angles; but an angle at the circumference being half of that at the centre, the vertical angle of the isosceles triangle, formed in the pentagon by drawing straight lines from any corner to the extremities of the opposite side, must also be the tenth part of four right angles. Whence the construction of a regular pentagon or decagon involves the description of an isosceles triangle, whose vertical angle is equal to the tenth part of four right angles, or the fifth part of two right angles.

2. Since the vertical angle of that isosceles triangle is the fifth part of two right angles, the angles at its base must be together equal to the remaining four-fifths, and each of them is consequently two-fifths of two right angles. Wherefore each of the angles at the base of that component triangle is double of its vertical angle.

3. Let ABC be such an isosceles triangle, having each of the angles at A and C double of the angle at B. Draw CD bisecting the angle ACB. The angle BCD must then be equal to CBD, and consequently the side CD is equal to BD. But in the triangles BAC and CAD, the angle ABC is equal to ACD, the angle CAB common to both, and consequently the remaining angle BCA is equal to CDA; whence CDA is equal to CAD, and therefore the side AC is equal to CD. Thus the three straight lines AC, CD, and BD are all equal. Again, because CD bisects the angle ACB, (VI. 11. El.) BC : AC : : AC : AD, that is, AB : BD : : BD : AD. Hence AB is divided in extreme and mean ratio at the point D,—or the square of BD or AC, the base of the isosceles triangle, is equal to the rectangle under the side AB and the remaining segment AD. Whence the construction of a regular pentagon or decagon, depends on the medial section of a straight line.



4. Now let the straight line AB be divided by a medial section, or $BC^2 = BA.AC$. Add to each the rectangle BA.BC, and $BC^2 + BA.BC = BA.AC + BA.BC$, or $BC(BA+BC)=BA^2$. To AB annex BD equal to it, and $BC.CD=BD^2$. Bisect BD in E, and the straight lines CD and BC are the sum and difference of CE and BE; whence the rectangle under CD and BC, or the square of BA, is



equal to the excess of the square of CE above the square of BE, and therefore $CE^2 = BA^2 + BE^2$. Erect the perpendicular $BF = BA$, and join EF. It is evident that, $EF^2 = BA^2 + BE^2$; and consequently $EF^2 = CE^2$, and $EF = CE$; but EF being given, CE and BC are therefore given.

The composition of this general problem forms a series of the most interesting propositions in elementary geometry. Art. 4. corresponds to Prop. 22. Book II.; Art. 3. to Prop. 3. and 4. Book IV.; and Art. 2. and 1. coincide with the 5th and 8th Propositions of the same Book.

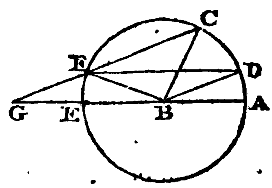
PROP. XXV. PROB.

To discover the conditions required for the trisection of an angle.

Let ABC be an angle, of which ABD is the third part. About the vertex B describe a circle, draw DF parallel to AB, join CF, and produce it to meet the extension of AB in G.

ANALYSIS.

Because the chord DF is parallel to AE, the arc EF (III. 20. El.) is equal to AD, and consequently (III. 13. cor. El.) the angle EBF is equal to ABD, or is half of the remaining angle DBC; but half this angle is equal (III. 17. El.) to the angle DFC at the circumference, and which (I. 23. El.) is equal to its opposite



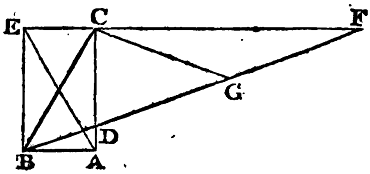
angle BGF. Wherefore the angles BGF and GBF are equal, and (I. 12. El.) the triangle BFG is isosceles; and thus the solution of the problem would require, to draw CFG, such that the extreme part FG shall be equal to the radius of the circle.

Otherwise thus.

Let the angle ABD be the third part of ABC. Erect the perpendicular ADC, complete the rectangle BACE, extend the side EC to meet BD produced in F, and draw CG making the angle FCG equal to CFG.

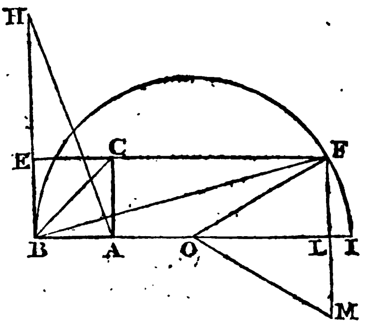
ANALYSIS.

Because the angle FCG is equal to CFG, the side GF (I. 12. El.) is equal to GC, and the exterior angle CGB (I. 32. El.) is double of either of those angles. But the angle CBA being triple of ABD, the angle CBG is double of ABD, or of CFG, and is therefore equal to CGB; whence the side BC is equal to GC. Again, from the right angles EBA and FCD, take away the equal angles ABD and FCG, and the remaining angles EBD and GCD are equal; but EBD is equal (I. 23. El.) to the alternate angle BDA, which is equal to the vertical angle CDF; consequently the angle GCD is equal to GDC, and therefore the side GD is equal to GC. Thus it appears, that the four straight lines BC, GC, GD, and GF, are all equal. Whence DF, the external segment of the trisecting line BF, is double of BC the diagonal of the rectangle BACE.



Scholium. Such then are the final conditions on which the trisection of an angle is made to depend. But to fulfil them in general, exceeds the powers of elementary geometry. In some very limited cases indeed, the trisection of an angle can be effected, merely by the help of straight lines and circles. Thus, when the proposed angle ABC is half a right angle, it may be trisected by the application of Prop. 21. For, pro-

duce BE so that $BH = 2BC$, join AH, produce BA till $AI = AH$, and on BI describe a semicircle meeting the production of EC in F; the angle ABF is the third part of ABC.



This result agrees with what is derived from simpler views. For $BH^2 = 4BC^2 = 8BA^2$, and $AI^2 = BH^2 + BA^2 = 8BA^2 + BA^2 = 9BA^2$; whence $AI = 3BA$, the diameter $BI = 4BA$, and consequently the radius $OI = 2BA$. Let fall the perpendicular FL, and produce it equally on the other side, join OF and OM. The triangles OFL and MOL are evidently equal, and therefore OF, OM, and FM, are all equal to 2BA, or 2FL; consequently the triangle FOM is equilateral, and the angle FOM two-thirds of a right angle; the angle FOL is hence one-third of a right angle, and the angle ABF at the circumference, being the half of it, is therefore equal to the sixth part of a right angle*.

PROP. XXVI. PROB.

To investigate the conditions required in finding two mean proportionals.

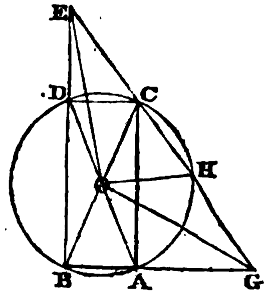
Let the containing sides AB and AC of the rectangle ABCD be the extremes of a continued proportion, of which the successive mean terms are DE and AG.

ANALYSIS.

Join CE and CG. Because AB or CD : DE : : AG : AC, and CDE, being a right angle, is equal to GAC, the triangles DCE and AGC are (VI. 14. El.) similar; whence the angle DEC is equal to ACG, and the angles ACG and ACE

* See Note LIII.

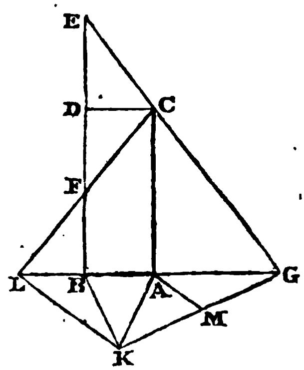
equal to DEC and ACE, or (I. 23. El.) two right angles, and consequently ECG forms a straight line. Draw the diagonals BC, AD, and join their intersection O with the points E and G. The triangles BOD and BOA being (I. 29. cor. El.) isosceles, therefore (II. 23. cor. El.) $OE^2 = OD^2 + BE.ED$ and $OG^2 = OA^2 + BG.GA$; but (VI. 12. El.) BG : BE : : GA : AC, or DE : GA, and hence (V. 6. El.) BE.DE=BG.GA. Wherefore, OD being equal to OA, the square of OE is equal to that of OG, and consequently the point O is equidistant from E and G. Hence, likewise, if a circle were described about the given rectangle, the intercepted segment EC (IV. 4. cor. El.) would be equal to GH.



The solution of the problem, then, requires to draw ECG, such that the distance OE be equal to OG, or that the part EC without the circle be equal to the opposite part GH.

Otherwise thus.

The first part of the construction remaining the same, it was proved that the rectangle BE.ED is equivalent to BG, GA; bisect BD in F, and $BE.ED + DF^2$, or (II. 19. cor. 2. El.) $EF^2 = BG.GA + DF^2$. On AB construct the isosceles triangle BKA, having each of its sides BK and AK equal to DF, and join GK; then (II. 23. cor. El.) $BG.GA + AK^2 = GK^2$, and consequently $EF^2 = GK^2$, or EF= GK. But, by hypothesis, AB : DE : : DE : : GA : : GA : AC,

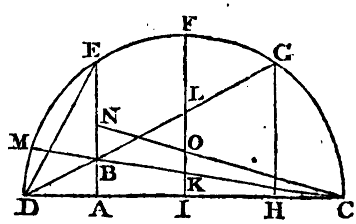


and (V. 16. El.) AB : GA : : DE : AC, or (V. 13. El.) 2AB : GA : : 2DE : AC; join CF and produce it to meet the extension of AB in L; the triangles CFD and LFB (I. 21. El.) are evidently equal, and CD or AB=BL. Wherefore AL is to GA as 2DE to AC or BD, or (V. 3. El.) as DE to DF the half of BD, and consequently (V. 9. El.) GL : GA : : EF : DF. Join LK and draw AM parallel to it; then (VI. 1. El.) GL : GA : : GK : GM, whence EF : DF : : GK : GM; but EF=GK, and therefore DF=GM. Now the points F, L and K are evidently given, and consequently the straight line LK and its parallel AM are given in position.

To effect, therefore, the construction of the problem, it is required from the point K to draw the straight line KMG, such that the part MG, intercepted between AM and BA produced, shall be equal to the half of AC.

Or thus.

Let AB and AC, the extreme terms of the continued proportion, stand as before at right angles, and having produced CA to D, let AB : AD : : AD : AE : : AE : AC. Since, then, AD : AE : : AE : AC, it follows (V. 6. El.) that AD.AC= AE^2; whence (III. 32. cor. 1. El.) the point E lies in the circumference of a semicircle described upon CD. Join DE, produce DB to the circumference, and draw the perpendicular radius IF. Because AB : AD : : AD : AE, and the angle DAE is common to the two triangles BAD and DAE—these triangles (VI. 14. El.) are similar; consequently the angle ADB is equal to AED, and (III. 18. cor. El.) the arc CG is equal to DE;



whence the arc FG is equal to FE, and (III. 13. and 4. El.)

the segment IH of the diameter equal to IA, or the oblique line GL (VI. 1. El.) is equal to LB.

On this condition therefore, that GD shall have its intercepted portion GL equal to LB, or that the perpendiculars EA and GH shall be equidistant from the centre, the solution of the problem depends. The ratio of KI to IC is evidently the same as that of AB to AC. Wherefore a semicircle being described with the radius IC—could a straight line BD be drawn from D, such that the part BG, intercepted between the circumference and the straight line CKM drawn from the other extremity of the diameter, be bisected in L by the perpendicular radius IF—the problem would be solved: For make AN=AD, and join CN meeting IF in O; it is manifest, from what has been shown, that IK, IO, IL, and IC are continued proportionals *.

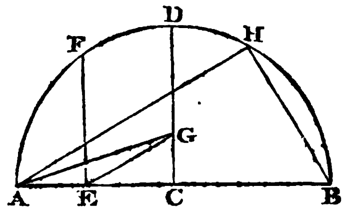
PROP. XXVII. THEOR.

If, from the extremity of the diameter of a circle, a straight line be drawn to a point in the perpendicular radius, such that triple its square be equal to the square of a perpendicular from the circumference and the squares of the segments into which the diameter is thus divided; the straight line that joins the points of section and of termination, will make a given angle with the diameter.

In the semicircle ADB, EF and the radius CD being at right angles to AB, and AG drawn so that $3AG^2 = AE^2 + EF^2 + EB^2$; if EG be joined, the angle CEG is given.

* See Note LIV.

ANALYSIS.



For join CF. Because AB is bisected in C, $AE^2 + EB^2 = 2AC^2 + 2EC^2$ (II. 21. cor. El.) and consequently $3AG^2 = 2AC^2 + 2EC^2 + EF^2$; but (II. 11. El.) $EC^2 + EF^2 = CF^2$, or AC^2, and hence $3AG^2 = 3AC^2 + EC^2$. Again, $AG^2 = AC^2 + CG^2$, or $3AG^2 = 3AC^2 + 3CG^2$; wherefore $EC^2 = 3CG^2$, or $EG^2 = 4CG^2$ and $EG = 2CG$. The ratio of EG to CG, and the right angle at C being thus given, the triangle EGC is (VI. 15. El.) given in species, and consequently the angle CEG is given.

COMPOSITION.

Inflect BH equal to the radius of the circle, join AH, draw EG parallel to it meeting CD in G, and join AG; then $3AG^2 = AE^2 + EF^2 + EB^2$.

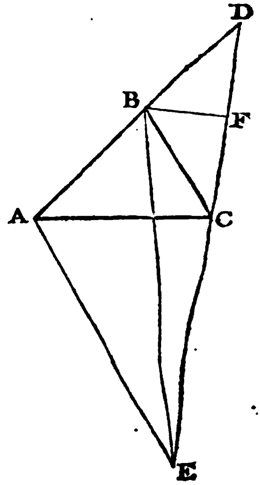
For join CF. The triangles AHB and ECG being evidently similar, AB : BH : : EG : CG; but AB=2BH, and therefore (V. 5. El.) EG=2CG. Whence $EG^2 = 4CG^2$, and $EC^2 = 3CG^2$; consequently $3AG^2 = 3AC^2 + 3CG^2 = 3AC^2 + EC^2 = 2AC^2 + 2EC^2 + AC^2 - EC^2$. Now $2AC^2 + 2EC^2 = AE^2 + EB^2$, and $AC^2 - EC^2 = EF^2$; wherefore $3AG^2 = AE^2 + EF^2 + EB^2$.

PROP. XXVIII. THEOR.

If a triangle have a given angle, the excess of the square of the sum of the containing sides above the square of the base, has a given ratio to the area of the triangle.

Let ABC be a triangle, in which AB is produced till BD be equal to BC; the excess of the square of AD above the square of AC, has a given ratio to the area of the triangle.

ANALYSIS.



Draw AE parallel to BC, and meeting DC produced in E, from B let fall the perpendicular BF, and join BE.

The triangle CBD being isosceles, the angle CDB (I. 11. El.) is equal to DCB, but (I. 23. El.) DCB is equal to CEA; hence the angles EDA and DEA are equal, and the triangle DAE is isosceles. Wherefore (II. 23. El.) $AD^2 = AC^2 + DC.CE$, or $AD^2 - AC^2 = DC.CE$. Again, because AE is parallel to BC, the triangle ABC has (II. 1. El.) the same area as EBC, or (II. 6. El.) is half the rectangle BF.CE. Consequently the excess of the square of AD above the square of AC, is to the area of the triangle ABC, as DC.CE to $\frac{1}{2}$ BF.CE, that is, (V. 23. cor. 2. El.) as DC to $\frac{1}{2}$ BF, or (V. 3. El.) as 4DF to BF. But the given angle ABC, being (I. 32.) equal to the two angles CDB and BCD, is double of either, and thus the angle BDF is given; whence the right-angled triangle DFB is given in species, and therefore the ratio of DF to BF is given. It thence follows, that the ratio of 4DF to BF, or that of the excess of the square of AD above the square of AC to the area of the triangle ABC, is given.

COMPOSITION.

The same construction remaining, DC.CE : BF.CE : : DC : BF; but $DC.CE = AD^2 - AC^2$, and BF.CE is double of the triangle ABC; whence 2DC is to BF, as the excess of the square of AD or that of the sum of the sides AB and BC above the square of the base AC, to the area of the triangle ABC *.

* See Note LV.

GEOMETRICAL ANALYSIS.

BOOK II.

DEFINITION.

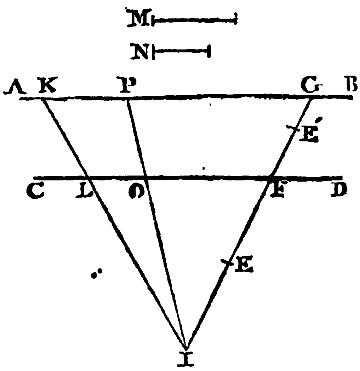
A VARIABLE quantity derived from another given or constant quantity, or which depends on it by some relation according to a given law, is necessarily confined between certain extreme limits. When it has acquired the greatest possible expansion, it is said to have reached its *maximum*; and when it has contracted into its lowest dimensions, it occupies the state of *minimum*.

PROP. I. PROB.

From a given point, to draw a straight line intercepting, on two given parallels, segments which shall have a given ratio.

Let AB and CD be two parallels, in which are two given points, P and O; and let it be required, from another given point E, to draw EF, such that PG shall be to OF in the ratio of M to N.

ANALYSIS.



Join PO, and produce it to meet EF, or its extension in I.

Because PG and OF are parallel, PI : OI : : PG : OF (VI. 2. El.); but the ratio of PG to OF is given, and hence that of PI to OI, and of PO to OI, are given. And since PO is given, OI and the point I, are given; wherefore IEF, and the segments PG and OF are given.

COMPOSITION.

Make PK=M and OL=N, join KL, PO, and produce them to meet in I, and draw IEF; PG and OF are the required segments.

For (VI. 2. El.) the parallels AB and CD being cut proportionally by the diverging lines IK, IP, and IG,—PG is to OF as KP to OL, that is, as M to N.

If M be equal to N, the point I vanishes, and EF becomes evidently a parallel to OP.

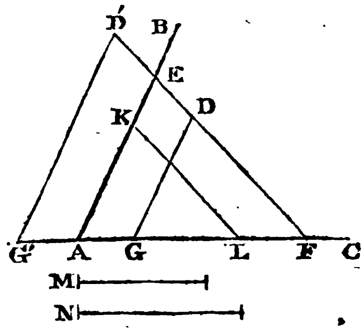
If the straight lines KL and PO meet in the given point E, the problem is by its nature indeterminate, or it admits of indefinite solution; for, in that case, the segments PG and OF, intercepted by any straight line whatever, drawn through E, have all the same ratio.

PROP. II. PROB.

Two diverging lines being given in position, to draw, through a given point, a straight line intercepting segments which shall have a given ratio.

Let it be required, through D, to draw EDF, so that AE shall be to AF in the ratio of M to N.

ANALYSIS.



Through D, (I. 24. El.) draw DG parallel to AE, and meeting AC, or its production, in G.

The triangles EAF and DGF are similar, and therefore (VI. 12.) AE : AF : : GD : GF; but the ratio of AE to AF is given, and consequently that of GD to GF. And since GD and the point G are evidently given, GF and the point F are likewise given.

COMPOSITION.

From AB and AC cut off AK = M, and AL = N, join KL, and parallel to it draw EDF through D; AE and AF are the segments required.

For (VI. 1. El.) the parallels EF and KL cut the diverging lines AB and AC proportionally, and therefore AE is to AF, as AK to AL, that is, as M to N.

PROP. III. PROB.

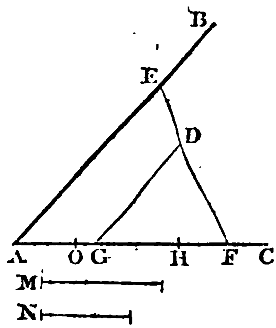
Two diverging lines being given in position, to draw, through a given point, a straight line cutting off segments—on the one from their intersection, and on the other from a given point — that shall have a given ratio.

Let AB and AC be two diverging lines, it is required, through the point D, to draw EDF, so that AE shall be to the part OF, in the ratio of M to N.

ANALYSIS.

Draw DG parallel to AE, and meeting AC, or its production in G, and make AE : GD : : OF : OH.

By alternation, AE : OF : : GD : OH; but the ratio of AE to OF is given, and thence that of GD to OH; and since GD and the point O are given, OH and the point H are also given. Again, because AE : GD : : OF : OH, and (VI. 2. El.) AE : GD : : AF : GF, it follows that OF : OH : : AF : GF; whence (V. 10. El.) FH : OH : : AG : GF, and (V. 6. El.) GF.FH=AG.OH. But AG and OH are both given, and consequently the rectangle under the segments GF and FH of the given portion GH is also given, and thence the point of section F is given, and the straight line ED.



COMPOSITION.

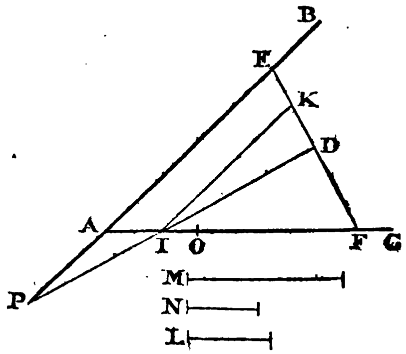
Make GD to OH, as M to N, and (VI. 19.) divide GH in F, so that the rectangle GF, FH shall be equal to AG.OH, and draw EDF; then the segment AE is to OF as M to N. Since GF.FH=AG.OH, therefore FH : OH : : AG : GF, and (V. 9. El.) OF : OH : : AF : GF : but (VI. 2. El.) AE : GD : : AF : GF, and consequently AE : GD : : OF : OH, and alternately AE : OF : : GD : OH, that is, in the given ratio.

PROP. IV. PROB.

Two diverging lines being given in position, to draw, through a given point, a straight line, cutting off segments from given points in a given ratio.

Let AB and AC be two diverging lines; it is required, through the point D, to draw EDF, so that PE shall be to OF in the ratio of M to N.

ANALYSIS.



Join DP cutting AF in I, and, through I, draw IK parallel to AB, and meeting EF in K.

Because the points D and P are given, the straight line DP is given in position, and consequently its intersection I with AC is given, whence IK, being parallel to AB, is likewise given in position. But (VI. 2. El.) PE : IK : : PD : ID, and since PD and ID are both given, the ratio of PE to IK is given; consequently, the ratio of PE to OF being given, the ratio of IK to OF is given. Wherefore, by the last proposition, the straight line KDF is given in position.

COMPOSITION.

Join PD and draw IK parallel to AB, make M to L, as PD to ID, and draw, by the last proposition, KDF, so that IK shall be to OF, as L to N; then will PE and OF be the segments required.

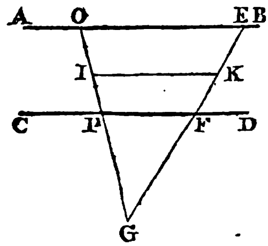
For (VI. 2. El.) PE : IK : : PD : ID : : M : L, and IK : OF : : L : N; whence (V. 16. El.) PE : OF : : M : N.

PROP. V. PROB.

Two parallels being given, from a point in a given intersecting line, to draw another straight line cutting off segments which shall contain a given rectangle.

Let AB, CD be two parallels, and G a given point, through which it is required to draw FE intercepting, from given points O and P in the same direction OPG, segments OE and PF, that shall contain a given rectangle.

ANALYSIS.



Because AB and CD are parallel, GO : GP : : OE : PF (VI. 2. El.) and consequently (V. 25. cor. 2. El.) GO : GP : : OE^2 : OE.PF; and GO and GP being given, their ratio is given, and therefore the ratio of OE^2 to OE.PF is given; but the rectangle OE, PF is given, and hence the square of OE and consequently OE itself, are given.

COMPOSITION.

Find (VI. 18. El.) GI, a mean proportional between GO and GP, draw IK parallel to AB or CD, and such (III. 33. El.) that its square shall be equal to the given rectangle, and join EKFG; this is the straight line required.

For OE, IK, and PF being parallel, OG : IG : : OE : IK, and PG : IG : : PF : IK (VI. 2. El.); whence compounding these analogies (V. 22. El.) OG.PG : IG^2 : : OE.PF : IK^2; but OG.PG$=IG^2$, and consequently (V. 4.) OE.PF$=IK^2$.

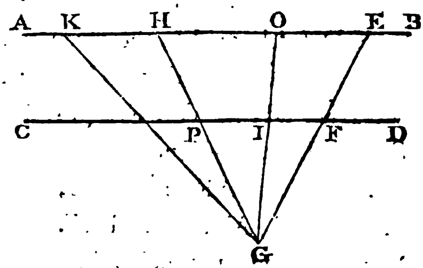
PROP. VI. PROB.

Through a given point, to draw a straight line intercepting, from given points on two given parallels, segments which shall contain a given rectangle.

Let AB and CD be parallels in which the points O and P are given, and let it be required through G to draw GFE, so that the segments OE and PF shall contain a given rectangle.

ANALYSIS.

Draw GO and GP, cutting the parallels in I and H. Because the points O, P, and G are given, the straight lines GIO and GPH are given in position, and consequently their intersections I and H with the given parallels. And since AB is parallel to CD, GP : GH : : PF : HE (VI. 2. El.) but (V. 25. cor. 2. El.) PF : HE : : PF.OE : HE.OE, and consequently GP : GH : : PF.OE : HE.OE. Now, GP and GH being given, their ratio is



given, and hence that of PF.OE to HE.OE; wherefore the rectangle PF, OE being given, the rectangle under the segments HE and OE of the given straight line HO is likewise given; whence (VI. 19. El.) the point E is given, and consequently the straight line GFE.

COMPOSITION.

Draw GO and GP, find (II. 9. El.) HK the side of a rectangle GP, HK which is equal to the given space, and (VI. 19. El.) divide HO in the point E, so that the rectangle under its segments HE and OE shall be equal to the rectangle HG, HK, and join GFE; this is the straight line required.

For HE : PF : : HG : GP, and hence (V. 25. cor. 2. El.) HE.OE : PF.OE : : HG.HK : GP.HK; but, by construction, the rectangle HE.OE is equal to GH.HK, and consequently (V. 8. and 4. El.) PF.OE = GP.HK, or the given space.

PROP. VII. PROB.

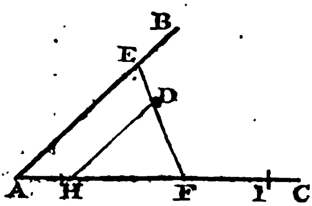
To draw through a given point a straight line, cutting from two given diverging lines, segments which shall contain a given rectangle.

Let AB and AC be two diverging lines given in position, and let it be required from the point D, to draw DFE, so that the rectangle under the segments AE, AF shall be equal to a given space.

ANALYSIS.

Draw HD parallel to AB, and make (II. 9. El.) the rectangle DH.AI equal to the given space.

Because AE.AF=DH.AI, AE : DH : : AI : AF (V. 6. *El.*), but AE : DH : : AF : FH (VI. 2. El.), and therefore AF : FH : : AI : AF; whence (V. 11. cor. El.) AH : AF : : IF : AI, and (V. 6. El.) AH.AI=AF.IF. Now DH, being parallel to AB, is given, and consequently AI is given; wherefore the rectangle AH, AI being given, AF.IF is also given; and since AI is given, its internal or external section is (VI. 19. El.) given.



COMPOSITION.

Draw DH parallel to AB, find (II. 9. El.) AI, which contains with DH a rectangle equal to the given space, and divide AI (VI. 19. El.) so that the rectangle under its segments AF, FI shall be equal to the rectangle AI, AH; EDF is the straight line required. For, by construction, AF.IF=AI.AH, whence (V. 6. El.) AH : AF : : IF : AI, and (V. 10. and 7. El.) AF : FH : : AI : AF; but AF : FH : : AE : DH, and consequently AE : DH : : AI : AF, and (V. 6. El.) AE.AF= DH.AI.

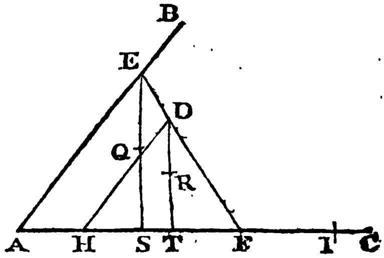
PROP. VIII. PROB.

Through a given point to draw a straight line, which shall, by its intersection with two given diverging lines, form a triangle containing a given space.

Let it be required, through the point D, to draw a straight line, EDF intercepting, between the diverging lines AB and AC, a triangle AEF, which shall contain a given space.

ANALYSIS.

Draw DH parallel to AB, upon AC let fall the perpendiculars ES and DT, and find (II. 9. and 7. El.) AI the base of a triangle, having the altitude DT and containing the given space.



Because the rectangles ES.AF and DT.AI are (I. 6. El.) each double of the triangles AEF and ADI, they are equal, and consequently (V. 6. El.) ES : DT : : AI : AF. But the triangles AES and HDT are evidently similar, and therefore (VI. 12. El.) AE : ES : : HD : DT, or alternately AE : HD : : ES : DT; whence AE : HD : : AI : AF, and AE.AF=HD.AI. Now HD is given, and consequently AI; wherefore the rectangle AE.AF is given, and thence, by the last proposition, the straight line EDF is given in position.

COMPOSITION.

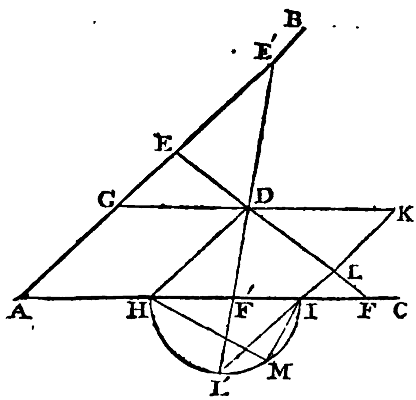
Draw DH parallel to AB, let fall the perpendicular DT, bisect this in the point R, find (II. 9. El.) the side AI, which with RT contains a rectangle equal to the given space, and, by the last proposition, draw EDF, such that the rectangle AE.AF shall be equal to DH.AI.

Having let fall the perpendicular ES, and bisected it in Q, the triangles AES and HDT are similar; whence AE : ES : : HD : DT, and alternately AE : HD : : ES : DT, or (V. 3. El.) AE : HD : : QS : RT; wherefore AE.AF : HD.AI : : QS.AF : RT.AI; but the rectangle AE.AF=HD.AI, and

hence (V. 4. El.) QS.AF=RT.AI, or the triangle AEF is equal to the given space.

This problem will admit of a simpler construction, in the case where the given point D lies between the diverging lines AB and AC. For draw DG parallel to AC, and make (II. 9. El.) the rhomboid AGKI equal to the given space.

Because the triangle AEF is equal to the rhomboid AGKI, take away from both the figure AEDKLF, and the triangles GED and ILF remain equal to the triangle DLK; but these supplementary triangles, being formed by parallel lines, are evidently similar, and consequently the homologous sides GD and IF are (VI. 28. El.) sides of a right-angled triangle, of which DK is the hypotenuse; wherefore (II. 11. El.) $GD^2+IF^2=DK^2$, or (I. 27. El.) $IF^2=HI^2-AH^2$. And since HI and AH are both given, it follows that IF is given.



COMPOSITION.

Construct the rhomboid AGKI equal to the given space, draw DH parallel to AB, on HI describe a semicircle, in which inflect HM equal to AH, join IM, and make IF, or IF′, equal to it; EDF, or E′DF′, is the base of the required triangle.

For (III. 22. El.) HMI being a right angle, $IH^2=HM^2+IM^2$ (II. 11. El.), or $DK^2=GD^2+IF^2$; whence (VI. 23. cor. 1. El.) the triangle DLK, or DLK′, is equal to the triangles GED and ILF, or to GE′D and IL′F; and, adding

to both the excess of the rhomboid AK above the triangle DLK, or DL′K′, the rhomboid AK is equal to the triangle AEF or AE′F′, which is, therefore, equal to the given space.

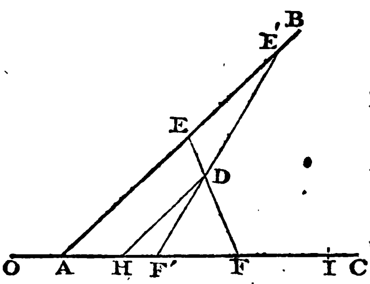
PROP. IX. PROB.

Through a given point to draw a straight line, cutting off segments, from two given diverging lines—on the one from their intersection, and on the other from a given point—which shall contain a given rectangle.

Let it be required to draw EDF, so that the rectangle AE, OF shall be equal to a given space.

ANALYSIS.

Draw DH parallel to AB, and (II. 9. El.) make the rectangle DH.OI equal to the given space; OI and the point I are, therefore, given. And since AE.OF = DH.OI, it follows that AE : DH : : OI : OF; but (VI. 2. El.) AE : DH : : AF : FH, and consequently AF : FH : : OI : OF. Wherefore (V. 11. El.) AF : AH : : OI : FI, and (V. 6. El.) AF.FI = AH.OI; hence AI and the rectangle under its segments, AF and FI, are given, and consequently (VI. 19. El.) the point of section F and the straight line EDF are given.



COMPOSITION.

Having drawn DH parallel to AB, and made the rectangle DH.OI equal to the given space, divide AI (VI. 19. El.) in F, or F′, such that the rectangle under its segments shall

also be equal to the rectangle AH.OI; EDF, or E′DF′, is the required straight line. For since AF.FI=AH.OI, AF : AH : : OI : IF; whence (V. 11. El.) AF : FH : : OI : OF; but (VI. 2. El.) AF : FH : : AE : DH, and, therefore, AE : DH : : OI : OF, and the rectangle AE.OF is equal to DH.OI, or the given space.

PROP. X. PROB.

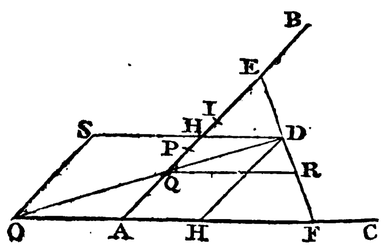
Through a given point, to draw a straight line, cutting off segments from given points, on two given diverging lines, that shall contain a given rectangle.

Let it be required to draw EDF, so that the rectangle OF.PE shall be equal to a given space.

ANALYSIS.

Join DO meeting AE in Q, and draw QR parallel to AC.

Because (VI. 2. El.) DO : DQ : : OF : QR, it follows (V. 25. cor. 2. El.) DO : DQ : : OF.PE : QR.PE; but DO and DQ are evidently given, and therefore the rectangle OF.PE has to QR.PE a given ratio; and since OF.PE is given, the rectangle QR.PE is likewise given, and QR, being parallel to AC, is given in position. Whence, by the last proposition, the intersecting line EDR or EDF, is given in position.



COMPOSITION.

Join DQO, draw DH parallel to AC, and produce it meet-

ing in S the parallel to AB, make the rectangle DS.PI equal to the given space, and divide PI in E, such that the rectangle under its segments PE, IE shall be equal to the rectangle AH.PI; EFD is the straight line required.

For DQ : DO : : DH : DS : : QR : OF, and consequently (V. 25. cor. 2. El.) DH.PI : DS.PI : : PE.QR : PE.OF; but, by the last proposition, DH.PI=PE.QR, whence the rectangle DS.PI, or the given space, is equal to the rectangle PE.OF.

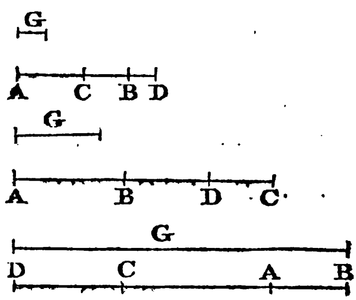
PROP. XI. PROB.

To divide a given straight line, so that the rectangle under one of its segments and a given line, shall be equal to the square of the other segment.

Let it be required to divide AB in C, such that the rectangle under AC and G shall be equal to the square of CB.

ANALYSIS.

Make BD=G, and since AC.G=CB^2, it follows (V. 6. El.) that AC : CB : : CB : BD; and consequently (V. 9. and 10. El.) AB : CB : : CD : BD; whence (V. 6. El.) AB.BD = CB.CD. But the rectangle AB.BD is given, and, therefore, the rectangle CB.CD is also given; and BD being given, the point of section C is (VI. 19. El.) thence given.



COMPOSITION.

In the same straight line AB, make BD equal to G, and (VI. 19. El.) cut BD such that the rectangle CB.CD be equal to AB.BD; C is the point of section required. For it is

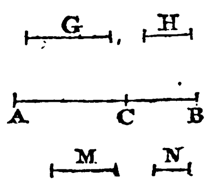
evident (V. 6. El.) that AB : CB : : CD : BD, and consequently (V. 10. El.) AC : CB : : CB : BD; wherefore (V. 6. El.) AC.BD, or AC.G$=$CB^2.

PROP. XII. PROB.

To divide a given straight line, so that the rectangle under one of its segments and a given line shall have a given ratio to the square of the other segment.

Let it be required to divide AB in C, such that AC.G : CB^2 : : M : N.

ANALYSIS.



Make (VI. 3. El.) G : H : : M : N, and H is given; but AC.G : : CB^2 : : G : H, and consequently (V. 25. cor. 2. El.) CB^2$=$AC.H ; wherefore, by the last proposition, the section of AB is given.

COMPOSITION.

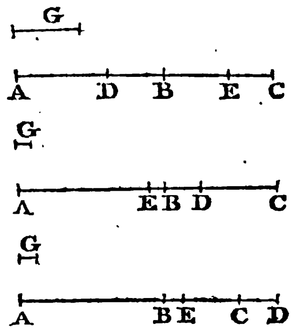
Having made M : N : : G : H, let AB be divided by the last proposition, so that AC.H$=$CB^2 ; then AC.G : CB^2 : : M : N. For AC.G : AC.H or CB^2 : : G : H, or M : N.

PROP. XIII. PROB.

In the same straight line, three points being given, to find a fourth point, such that the rectangle under its distance from the first and a given line, shall be equal to the rectangle under its distances from the second and third points.

Let it be required to find the point D, so that AD.G= CD.BD.

ANALYSIS.



Make BE=G, and because AD.G = CD.BD, it follows that AD : CD : : BD : BE; whence (V. 9. and 10. El.) AC:CD : : DE : BE, and AC.BE= CD.DE. But the rectangle AC.BE being evidently given, the rectangle under the segments CD, DE of CE, a given straight line, is also given, and consequently (VI. 19. El.) the point of section D is given.

COMPOSITION.

Having made BE=G, divide (VI. 19. El.) CE in D, so that CD.DE=AC.BE; D is the point required.

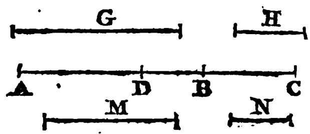
For (V. 6. El.) AC : CD : : DE : BE, and (V. 10. El.) AD : CD : : BD : BE; whence AD.BE, or AD.G= CD.BD.

PROP. XIV. PROB.

In the same straight line, three points being given, to find a fourth, so that the rectangle under its distance from the first and a given line, shall have a given ratio to the rectangle under its distances from the second and third points.

Let it be required to find a point D, such that AD.G : CD.BD : : M : N.

ANALYSIS.



Make M : N : : G : H, whence H is given; but since AD.G : CD.BD : : G : H, it is evident that AD.H = CD.BD; wherefore, by the last proposition, the point of section D is given.

COMPOSITION.

Having made G : H : : M : N, find, by the last proposition, the point D, so that CD.BD = AD.H; D is the section required. For (V. 25. cor. 2. El.) AD.G : AD.H or CD.BD : : G : H, or M : N.

PROP. XV. PROB.

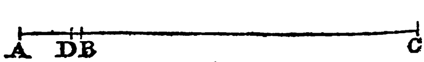
In the same straight line, three points being given, to find a fourth, so that the square of its distance from the first, shall be equal to the rectangle under its distances from the second and third points.

Let it be required to find a point D, such that AD^2 = CD.BD.

1. *When the point D lies between A and B.*

ANALYSIS.

Because AD^2 = CD.BD, it follows (V. 6. El.) that CD : AD : : AD : BD; whence (V. 9. El.) AC : AD : : AB : BD, and alternately AC : AB : : AD : BD.



But the ratio of AC : AB being given, the ratio of AD to BD is given; and since AB is given, the point D (VI. 4. El.) is given.

COMPOSITION.

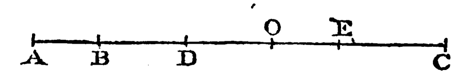
Divide AB (VI. 4. El.) in the ratio of AC to AB, and the point of section D is that required.

For, because AD : BD : : AC : AB, AD : BD : : AC—AD, or CD : AB—BD or AD (V. 19. cor. 1. El.); whence (V. 6. El.) AD^2=BD.CD.

2. *When the point D lies between B and C.*

ANALYSIS.

Make DE=AD, and since AD^2=CD.BD, CD : AD, or DE : : AD : BD, and therefore (V. 10. El.) CE : DE : : AB : BD, and alternately CE : AB : : DE : BD, or (V. 3. El.) CE : AB : : 2DE, or AE : 2BD; whence (V. 19. El.) CE : AB : : CE+AE or AC : AB + 2BD, or BE, and consequently (V. 6. El.)



CE.BE=AB.AC; but the rectangle AB.AC being given, the rectangle CE.BE is likewise given, and BC being given, the point E is given (VI. 19. El.), and therefore D, the bisection of AE, is given.

COMPOSITION.

Divide BC (VI. 19. El.) in E, such that CE.BE=AB.AC, and bisect AE in D; then AD^2=CD.BD.

For since CE.BE=AB.AC, it is evident that CE : AB : : AC : BE; whence (V. 19. El.) CE : AB : : AE : 2BD or (V. 3. El.) DE : BD; and alternately, CE : DE : : AB : BD, and (V. 9. El.) CD : DE, or AD : : AD : BD; wherefore (V. 6. El.) CD.BD=AD^2.

This last case is evidently subject to limitation; for the rectangle AB.AC being equal by construction to CE.BE, must not exceed the square of the half of BC, which (II. 19. cor. 1. El.) is the greatest rectangle contained under the segments of BC. Consequently, if E coincide with the middle

point O, it limits the problem; but then $AB.AC=BO^2$, or $AB.AC+BO^2=$(II. 23. cor. 2. El.) $AO^2=2BO^2$, and therefore AO is the diagonal of a square described on BO. Whence $AB : BC :: \sqrt{2}-1 : 2$, or $1 : 2+\sqrt{8}$; that is, the ratio of AB to BC has attained its *maximum*, when it is that of half the side of a square to the sum of the side and the diagonal.

PROP. XVI. PROB.

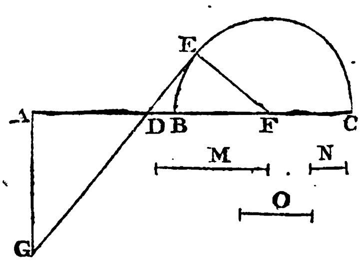
In the same straight line, three points being given, to find a fourth, such that the square of its distance from the first, shall have a given ratio to the rectangle under its distances from the second and third points.

Let it be required to find a point D, such that AD^2 shall be to CD.DB in a given ratio.

1. *When D lies between the points A and B.*

ANALYSIS.

On BC describe a semicircle, and draw the tangent DE; then (III. 32. cor. 2. El.) $DE^2=CD.DB$, and consequently the square of AD is to the square of DE in the given ratio; whence the ratio of AD to DE is given. Draw the radius EF, and produce ED to meet AG a perpendicular to AC. The triangles ADG and EDF are evidently similar, and therefore $AD : AG :: DE : EF$, or alternately $AD : DE ::$



$AG : EF$; and since the ratio AD to DE is given, the ratio of AG to EF is also given, and the radius EF being given, AG and the point G are thence given; wherefore the tangent GE and its intersection D with AC, are given.

COMPOSITION.

Let M : N be the given ratio, and to these find (VI. 18. El.) a mean proportional O, on BC describe a semicircle, make O : M : : BF : AG, a perpendicular erected from A, and (III. 26. El.) draw the tangent GDE; the intersection D is the point required.

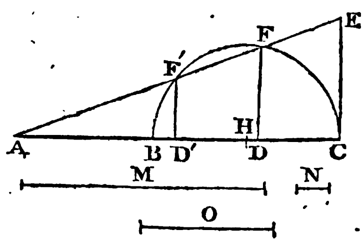
For, the triangles DAG, and DEF being similar, AD : AG :: DE : EF, and alternately AD : DE : : AG : EF, or M : O; wherefore (V 22. cor. 1. El.) $AD^2 : DE^2 :: M^2 : O^2$, that is, (V. 24. El.) M : N; but (III. 32. cor. 2. El.) $DE^2 = CD.DB$, and consequently $AD^2 : CD.DB :: M : N$.

2. *When D lies between the points B and C.*

ANALYSIS.

On BC describe a semicircle, draw DF perpendicular to the diameter, and meeting the circumference in F, and join AF.

Because (III. 32. cor. 2. El.) $BD.DC = DF^2$, the ratio of AD^2 to DF^2 is given, and consequently that of AD to DF; but the angle ADF, contained by these sides, being a right angle, is given, and therefore the triangle AFD is given in species. Hence the angle DAF is given, and the straight line AF given in position; wherefore the intersection F or F′, the perpendicular FD, or F′D′, and the point D, or D′, are all given.



COMPOSITION.

Let M : N express the given ratio, and to these find (VI. 18. El.) a mean proportional O, make (VI. 3. El.) M to O as AC to the perperdicular CE, join AE meeting the circumference of a semicircle described on BC in the point F or F′, and let

fall the perpendicular FD or FD'; then M : N :: AD^2 : BD.DC or AD'^2 : $BD'.D'C$.

For the triangle ACE is evidently similar to ADF or $AD'F'$, and therefore AC : CE :: AD : DF, and $AC^2 : CE^2 :: AD^2 : DF^2$; but (V. 24. El.) M : N :: $M^2 : O^2$, or as $AC^2 : CE^2$, and consequently $AD^2 : DF^2$, that is BD.DC, :: M : N.

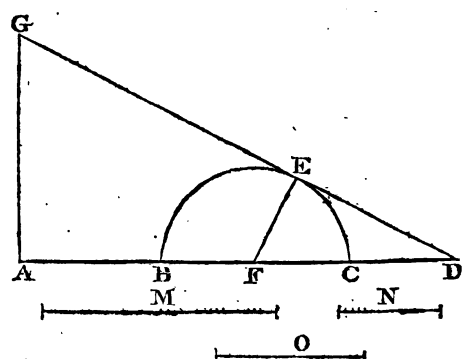
This problem evidently requires limitation; for, if AE should diverge too much from AC, it will not meet the circumference at all. Hence an extreme case will occur, when AE touches the circle. But the ratio of AC to CE, or of AD to DF, will then be the same as that of a tangent from A is to the radius HB; and consequently the limiting ratio is the duplicate of this,—or the ratio of M to N can never approach nearer to the ratio of equality than that of AB.AC, or $AH^2—HB^2$, to HB^2.

3. *When the point D lies beyond B and C.*

ANALYSIS.

On BC describe a semicircle, draw the tangent DE, and produce it to meet the perpendicular AG, and join E with the centre F.

Because (III, 32. cor. 2. El.) BD.DC$=DE^2$, the ratio of AD^2 to DE^2 is given, and consequently that of AD to DE. But the angle DEF, being (III. 24. El.) a right angle is equal to DAG, and the angle at D is common to the triangles DGA and DFE, which are therefore similar, and hence AD:AG :: DE : EF, or alternately AD : DE :: AG : EF. And



since the ratio of AD to DE is given, that of AG to EF is also given, and EF, the half of BC, being given, AG and the point G are thence given. Wherefore the tangent GE and its intersection D with AC, are given.

COMPOSITION.

Let M : N be the given ratio, and find the mean proportional O; make O : M : : BF : AG, a perpendicular to AC, and draw (III. 26. El.) the tangent GED; then M : N : : AD^2 : BD.DC.

For join EF. Because the triangles ADG and EDF are similar, AG : AD : : EF : ED, and alternately AG : EF : : AD : ED; but AG : EF : : M : O, and therefore M : O : : AD : ED, and $M^2 : O^2 :: AD^2 : ED^2$, that is, M : N : : $AD^2 : ED^2$ or BD.DC.

PROP. XVII. PROB.

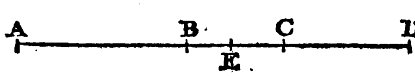
In the same straight line, four points being given, to find a fifth, such that the rectangle under its distances from the first and second points, shall have a given ratio to the rectangle under its distances from the third and fourth.

Let it be required to find a point E, so that AE.EB : DE.EC : : M : N.

1. *Let M : N be a ratio of equality.*

ANALYSIS.

Because AE.EB=DE.EC, it is manifest that AE : CE : : DE : EB; whence (V. 9. and 8. El.) AC : BD : : CE : EB, and (V. 9. El.)



AC+BD : BD : : BC : EB; but the ratio of AC+BD to BD is given, whence that of BC to EB, and therefore BE and the point E, are given.

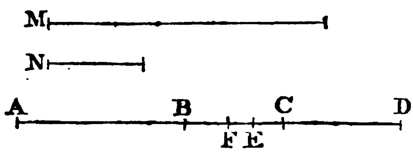
COMPOSITION.

Make AC+BD : BD :: BC : EB, and E is the point required. For (V. 10. El.) AC : BD :: CE : EB, and (V. 19. cor. 1. El.) AE : ED :: CE : EB, and hence (V. 6. El.) AE.EB=CE.ED.

2. *Let M : N be a ratio of majority or minority.*

ANALYSIS.

Find, by the preceding construction, a point F, such that AF.FB=DF.FC.



Because AE.EB : DE.EC :: M : N, it follows that AE.EB : AE.EB—DE.EC :: M : M—N ; but AE.EB—DE.EC=(AE.EB—AF.FB)+(DF.FC—DE.EC), that is, = EF (AF + BE) + EF (DF + CE), or = EF (AD+BC.) Wherefore AE.EB : EF (AD+BC) :: M : M—N; consequently the point E is assigned by Prop. 14. of this Book.

The composition of the problem is thence easily derived, by retracing the steps.

PROP. XVIII. PROB.

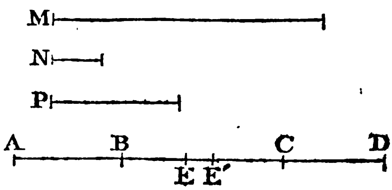
In the same straight line, four points being given, to find a fifth, such that the rectangle under its distances from the extreme points shall have a given ratio to the rectangle under its distances from the mean points.

Let it be required to find a point E, so that AE.ED : BE.EC :: M : N.

1. *Let AB=CD.*

ANALYSIS.

Because AE.ED = (AB + BE) (AB + EC), it is evident that AE.ED = AB.AC + BE.EC, whence AE.ED : AB.AC : : M : M—N. The ratio of AE.ED to AB.AC is therefore given, and the rectangle under AE and ED, the segments of AD, being thus given, the point E is assigned by VI. 19. of the Elements.



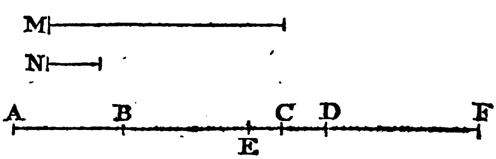
COMPOSITION.

Make M—N : M : : AB : P, and (VI. 19. El.) cut AD in E or E′, such that AE.ED = P.AC; E is the point required. For (V. 7. El.) M : M—N : : P : AB, and hence (V. 25. cor. 2. El.) M : M—N : : P.AC, or AE.ED : AB.AC; consequently M : N : : AE.ED : AE.ED—BA.AC, or BE.EC.

2. *Let AB and CD be unequal.*

ANALYSIS.

Because AE.ED = (BE + AB) (EC + CD) = BE.EC + BE.CD + AB.ED, consequently AE.ED—BE.EC = BE.CD + AB.ED = BD.CD — ED.CD + AB.ED = BD.CD + (AB—CD) ED. Produce AD to F, so that (AB—CD)DF = BD.CD; and since AB, CD and BD are all given, DF and the point F are given. Thus from construction AE.ED—BE.EC = (AB—CD) (DF + ED) = (AB—CD) EF. Now, since the



ratio of AE.ED to BE.EC is given, the ratio of AE.ED to EF (AB—CD) is also given; wherefore AB—CD being given, and the points A, C, and F, the point E is given by Prop. 14.

Applying that proposition, the construction of the problem is easily obtained.

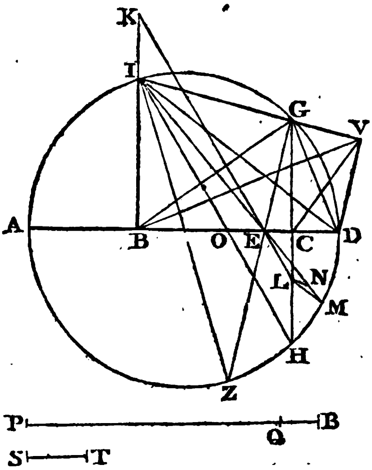
It yet remains to assign the limitations of this problem.

On AD describe a circle, erect the perpendiculars BI and GCH, join IOH, and, parallel to this, draw KEL through the point of section E, join OG, EG, and IE, which produce to the circumference, and join MG and ML.

The point O is evidently given. But the ratio of AE.ED to BE.EC may be considered as compounded of the ratio of AE.ED, or (III. 32. El.) IE.EM, to KE.EL, and of the ratio of KE.EL to BE.EC.

Now, since BK and CL are parallel, KE : EL : : BE : EC, or alternately KE : BE : : EL : EC, and therefore (V. 22. El.) $KE^2 : BE^2$: : KE.EL : BE.EC. Again, KE and IO being parallel, KE : IO : : BE : BO, or alternately KE : BE : : IO : BO, and hence $KE^2 : BE^2 : : IO^2 : BO^2$. Wherefore $IO^2 : BO^2$: : KE.EL : BE.EC, and consequently, the ratio of these rectangles is given; let it be that of PQ to ST.

The angle MGL, being equal (III. 18. El.) to MIH in the same segment, is equal (I. 23. El.) to the exterior angle MEL, and consequently (III. 18. cor. El.) the quadrilateral figure MGEL being thus contained in a circle, the angle LME is (III. 18. El.) equal to LGE. Draw LN making the angle MLN equal to EGO, and (I. 32. El.) the exterior angle LNE will be equal to CGO. But the triangles GOC and HOC are obviously equal, and, therefore, the angle CGO = CHO = CLE = EKI. Whence the triangles IEK and LEN are similar, and IE : KE : : EL : EN, and consequently



KE.EL=IE.EN. Make, therefore, PQ to PR, as EN to EM. The ratio of AE.ED to BE.EC is hence compounded of that of PR to PQ, and of PQ to ST, or it is the same with the *ratio* of PR to ST. But as the point of section E approaches to O, the angle EGO, or MLN, evidently diminishes, and consequently the point N, in a corresponding degree, approximates to M. Hence the extreme term which PR can never pass, is PQ; and therefore the limiting ratio of the rectangle AE, ED to BE, EC is that of PQ to ST, or of IO^2 to BO^2.

The point O of ultimate section, is hence easily determined. Because BI and CH are parallel, BI : CH : : BO : OC, and BI^2 or AB.BD : CH^2, or AC.CD : : BO^2 : OC^2. Wherefore BC must be divided (I. 14. Anal.) into segments BO and OC, which are in the subduplicate ratio of the rectangle AB, BD to AC, CD.

But the limiting ratio may be found in a more direct manner. For join IG, and draw DV perpendicular to it, join DG, DI, CV, and BV, and, having drawn the diameter IZ, join GZ. Because the angles DGC and DIV stand upon equal arcs DH and DG, they are equal (III. 18. El.); but the quadrilateral figures DCGV and DBIV, being right angled at B, at C and V, are each contained in a circle (III. 19. cor. El.); wherefore (III. 18. El.) the angle DGC is equal to DVC, and the angle DIV to DBV, and consequently the angles DVC and DBV are equal. Hence the triangles CDV and VDB, having besides a common vertical

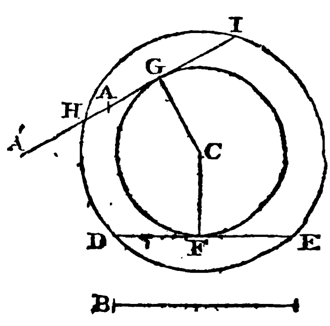
angle, are similar; and, therefore, BD : DV : : DV : DC, and (V. 6. El.) $BD.DC = DV^2$. But (VI. 16. cor. 1. El.) $DG^2 = AD.DC$, and consequently $DG^2 - DV^2$ or (II. 11. El.) $GV^2 = AD.DC - BD.DC$, or AB.DC. In the same manner, it is shown that $IV^2 = AC.DB$. Whence IG is given, being the difference between the sides of two squares that are equal to the rectangles AC, DB, and AB, DC. Again, the angle BIO, being equal to the alternate angle GHI, is equal (III. 18. El.) to GZI, and the right angle OBI is equal to the angle IGZ in a semicircle; wherefore the triangles IOB and ZIG are similar, and IO : BO : : IZ or AD : IG. Hence the limiting ratio of AE.ED to BE.EC, or that which marks the state of *minimum*, is the duplicate ratio of AD to the difference of the sides of squares equal respectively to the rectangle AC, DB and to the rectangle AB, DC.

PROP. XIX. PROB.

Through a given point, to draw a straight line, so that the part intercepted by the circumference of a given circle, shall be equal to a given straight line.

Let A be a point, through which it is required to draw a straight line HI, limited by a given circumference and equal to B.

ANALYSIS.



Take any point D in the given circumference, and inflect DE equal to B. Because DE is equal to B, it is equal to HI, and, therefore, (III. 11. El.) the chords HI, DE are equally distant from the centre of the circle, or CG=CF. But DE being given, CF is given, and thence the circle described from

through F and G; wherefore the point A being given, the tangent AG to that circle is given, and consequently HI is given in position.

COMPOSITION.

Inflect DE equal to B, from C let fall the perpendicular CF, with which distance describe a concentric circle, and draw (III. 26. El.) the tangent HAI.

It is evident that the chords HI and DE, being equidistant from the centre, are both of them equal to B.

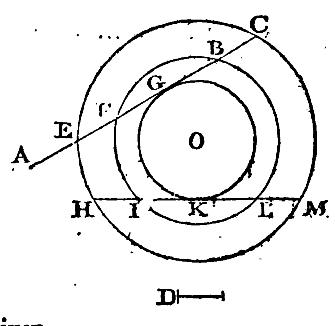
PROP. XX. PROB.

Through a given point, to draw a straight line, such that the part of it intercepted between two concentric circles shall be equal to a given straight line.

Let it be required, through the point A, to draw the straight line ABC, so that the part BC intercepted by the two concentric circles HECM and IFBL shall be equal to D.

ANALYSIS.

From any point H, in one of the circumferences, inflect HM=EC, and upon these let fall the perpendiculars OK and OG. The equal chords HM and EC are therefore equidistant from the centre, and reciprocally IL is equal to FB; consequently the halves of these are equal, or HK=GC, and IK=GB; whence the difference HI, being equal to BC, is given. But since the point H is given, the point I and the chord HM are given; and the circle which touches at K being given, the tangent AGC is also given.



COMPOSITION.

In the circumference of one of the circles, having assumed a point H, place HI equal to D, and produce it to M, upon this let fall the perpendicular OK, with which as a radius describe a circle, and apply to it the tangent ABC; then will the intercepted portion BC be equal to D.

For the chords EC and FB are (III. 11. El.) equal *to the* equidistant chords HM and IL; consequently *their halves* are equal, or GB=IK, and GC=HK, and *hence* BC=HI=D.

It is evident, that the interval BC between the *concentric* circles will be least when AC passes through the centre, and greatest when it touches the inner circle. Wherefore D is limited on both sides; not being less than the difference of the radii of the circles, nor its square greater than the difference of their squares.

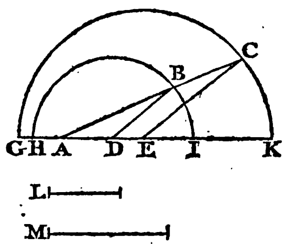
PROP. XXI. PROB.

Two circles described upon the same straight line being given, to draw from a point similarly placed in it another straight line, so that the part intercepted by the circumferences shall be equal to a given straight line.

Let D, E be the centres of the two circles, and let AD : AE : : DI : EK; it is required from A to draw ABC, such that BC shall be equal to L.

ANALYSIS.

Join BD and CE. Because AD : AE : : DI or DB : EK, or EC, therefore (VI. 1. cor. El.) DB is parallel to EC; whence AD : DE : : AB : BC, and since AD and DE are given, the ratio of AB to BC is given; but BC is given, and consequently AB is given, both in magnitude and position.



COMPOSITION.

Make (VI. 3. El.) EK—DI : DI : : L : M, and from A inflect AB equal to M; ABC is the straight line required.

For since, by hypothesis, AD : AE : : DI or DB : EK or EC, DB is parallel to EC; wherefore DB or DI : EC or EK : : AB : AC, and consequently (V. 11. cor. El.) EK—DI : DI : : BC : AB; but EK — DI : DI : : L : M or AB, whence BC : AB : : L : AB, and therefore (V. def. 10. El.) BC=L.

PROP. XXII. PROB.

Two circles described upon the same straight line being given, to draw, from the extremity of either diameter, another straight line, so that the part of it intercepted by the circumferences shall be equal to a given straight line.

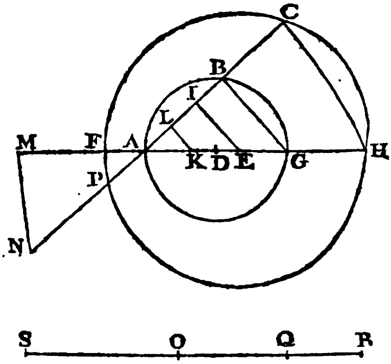
Let it be required to draw ABC, so that the intercepted portion BC shall be equal to QR.

ANALYSIS.

Join BG, CH, and FP, from E, the centre of the exterior circle, let fall upon AC the perpendicular EI, cut off IL=IB and draw LK parallel to BG, in the extension of AH make

(VI. 3. El.) AK : AG : : AF : AM, and, from the point M, draw MN parallel to FP, and meeting the production of AC.

Because LK is parallel to BG and FP to MN, therefore (VI. 1. El.) AK : AG : : AL : AB, and AF : AM : : AP : AN; but, by construction, AK : AG : : AF : AM, and, consequently, AK : AG : : AL : AB : : AP : AN. Whence (V. 19. El.) AK : AG : : AL + AP or PL : AB + AN or BN. Now, since (III. 4. El.) IP = IC, and IL = IB, therefore PL = BC or QR; and LK, IE, and BG being



parallel lines, KE = EG (VI. 1. El.) and thence AK is given; wherefore three terms of the analogy being given, the fourth term BN is given, and consequently BN + BC, or NC, is given. But the angle ACH is equal to AFP (III. 18. El.) which again (I. 23. El.) is equal to AMN, and hence the triangles CAH and ANM, having also the same vertical angle, are similar, consequently AH : AC : : AN : AM, and (V. 6. El.) AH.AM = AC.AN, wherefore NC and the rectangle under its segments AC, AN being given, AC is given in magnitude (VI. 19. El.) and hence likewise in position.

COMPOSITION.

Having cut off KE = EG, make (VI. 3. El.) AK : AG : : AF : AM, and AK : AG : : QR : QS, divide (VI. 19. El.) SR in O such that SO.OR = AH.AM, and inflect AC = OR; AC is the straight line required.

For join CH, BG, FP, and draw MN parallel to FP, and EI and KL parallel to CH. Since IL = IB and IP = IC, therefore PL = BC. The triangles CAH and ANM are si-

milar, AH : AC :: AN : AM, and AH.AM=AC.AN; but, by construction, AH.AM = SO.OR, and AC = OR, consequently AN = SO. Now, from the property of parallels, AK : AG :: AL : AB, and, by hypothesis, AK : AG :: AF : AM, or AP : AN; wherefore (V. 19. El.) AK : AG :: AL + AP or BC : AN + AB or BN. Whence BC : BN :: QR : QS, and (V. 11. El.) BC : CN :: QR : SR; but CN= SR, and consequently BC=QR.

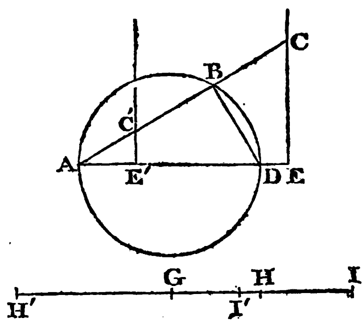
PROP. XXIII. PROB.

From the extremity of the diameter of a given circle, to draw a straight line, so that the part of it intercepted between a given perpendicular and the circumference, shall be equal to a given straight line.

Let it be required from A to draw AC, such that the intercepted portion BC shall be equal to GH.

ANALYSIS.

Join BD. The angle ABD, being in a semicircle, is a right angle, and therefore equal to AEC; consequently the triangles DAB and CAE, having besides a common angle at A, are similar, and AB : AD :: AE : AC, and hence AB.AC = AD.AE. But the rectangle AD,AE is given, and thence AB,AC;



and since BC is given in magnitude, therefore (VI. 19. El.) AB is given in magnitude, and consequently in position.

COMPOSITION.

Produce GH (VI. 19. El.) till GI.IH=AD.AE, and inflect IH from A to B; AB is the straight line required. For

join BD. The triangles ABD and AEC being evidently similar, AB : AD :: AE : AC, and consequently AB.AC= AD.AE=GI.IH; but AB=IH, whence AC=GI, and therefore BC=GH.

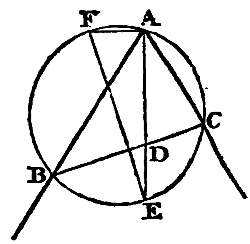
PROP. XXIV. PROB.

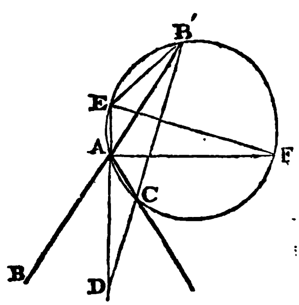
Through a given point in the line bisecting a given angle, to draw a straight line limited by the sides, and equal to a given straight line.

Let it be required, through the point D, situate in the straight line AD which bisects the angle BAC, to draw BC equal to a given straight line.

ANALYSIS.

About the points B, A and C, describe (III. 10. El.) a circle, draw the diameter EF, and join AF. Because BC and the angle BAC are given, the circumscribing circle (III. 27. El.) and consequently the triangle BAC, are given in magnitude: But since the angle BAE is equal to CAE, the arc BE is (III. 18. cor. El.) equal to CE; and hence the chord BC is bisected at right angles by the diameter EF. Wherefore AD being given, AE is, by the last proposition, given in magnitude, and thence DB is given in magnitude, and consequently in position.





COMPOSITION.

On the given straight line describe (III. 27. El.) a segment BAC, containing an angle equal to the given angle, and complete the circle, bisect the arc BAC in E, and from

that point draw, by the last proposition, EAD, such that AD shall be equal to the distance of the given point from the vertex; and DB,DC are the segments of the required line, from which its position is immediately determined.

For the angle BAC is equal to the given angle, and AD bisects it, since the arc BE=CE; but AD is besides equal to the distance of the given point from the vertex, and BC is equal to the given straight line. Wherefore all the points and lines retain, by this construction, their relative position.

Since AE cannot exceed the diameter FE, the limiting case will occur when these lines coincide; whence BC is the least possible when at right angles to AD, and therefore intercepting equal segments AB and AC.

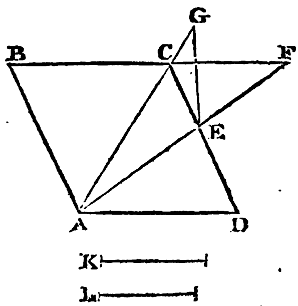
PROP. XXV. PROB.

Between the side of a given rhombus and its adjacent side produced, to insert a straight line of a given length, and directed to the opposite corner.

Let ABCD be a rhombus, of which the side BC is produced; it is required, from the opposite corner A, to draw AEF, such that the exterior portion EF shall be equal to a given straight line.

ANALYSIS.

Join AC, and, meeting this produced, draw EG, making the angle AEG equal to ACF.



The triangles CAF and EAG are evidently similar, and AC : CF :: AE : EG; but CE being parallel to AB, BC : CF :: AE : EF (VI. 1. El.); whence (V. 17. El.) AC : BC :: EF : EG. But AC, BC, and EF being given, EG is (VI. 3. El.) also given. Again, the angle ACD is (I. 2. El.)

equal to ACB, and therefore to FCG; consequently adding ECF to each, the whole angle ACF, or AEG, is equal to ECG. Hence the triangles AGE and EGC are similar, and AG : EG :: EG : GC, or $AG.GC = EG^2$. Wherefore the rectangle AG, GC is given, and consequently (VI. 19. El.) the point G, and thence the point E and the straight line AF.

COMPOSITION.

Let the intercepted segment be equal to K, join AC, make AC : BC : : K : L, divide AC in G (VI. 19. El.) so *that* $AG.GC = L^2$, and from G, with the radius L, describe a circle cutting CD in E; AEF is the straight line required.

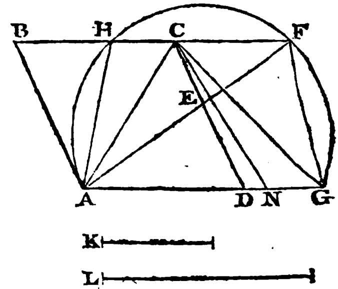
For since $AG.GC = L^2 = EG^2$, AG : EG : : EG : GC, and therefore the triangles AGE and EGC are similar, and the angle AEG is equal to ECG, or ACF; whence the triangles AFC and AGE are likewise similar, and AC : CF : : AE : EG; but (VI. 1. El.) BC : CF : : AE : EF, and consequently (V. 17. El.) AC : BC : : EF : EG. Now AC : BC : : K : L or EG; wherefore EF = K.

Otherwise thus.

ANALYSIS.

Draw FG making the angle AFG equal to ADC, cut off CH = CE, from C inflect CN = CA, and join CG and AH.

The triangle ACN being isosceles, the angle CAN is (I. 11. El.) equal to CNA; and since the diagonal AC bisects the angle BCD of the rhombus, the triangles ACE and ACH are (I. 3. El.) likewise equal, and hence AE is equal to AH, and the angle CAE equal to CAH. And because the triangles ADE and AFG are similar, AD : AE



: : AF : AG and AD.AG = AE.AF. But the angle ACD,

being equal to CAD, is equal to CNA, and consequently the triangles ADC and ACN are similar; whence AN : AC : : AC : AD, and therefore $AN.AD = AC^2$. Again, because AC bisects the vertical angle HAF (VI. 21. El.) $FA.AH = AC^2 + FC.CH$, that is, $FA.AE = AC^2 + FC.CE$; wherefore $FC.CE = FA.AE - AC^2$, that is, $AG.AD - AN.AD$, or NG.AD. But BA and CE being parallel, FC : EF : : AD : AE : : AF : AG, and CE : EF : : AB or AD : AF; consequently (V. 22. El.) $FC.CE : EF^2$: : AD : AG : : (V. 25. cor. 2. El.) NG.AD : NG.AG; since therefore $FC.CE = NG.AD$, it follows (V. 8. and 4. El.) that $EF^2 = NG.AG$. Hence (VI. 19. El.) AG and the point G are given, and the angle AFG, being equal to ADC, is (III. 27. El.) contained in a given segment of a circle; wherefore the intersection F and the inflected line AF, are given.

COMPOSITION.

Let K be equal to the intercepted portion of the straight line which is to be inflected from A, and find (II. 13. El.) L the side of a square equivalent to the squares of K and of the diagonal AC, produce AD, and from C place CG equal to L, upon AG describe (III. 27. El.) a segment of a circle containing an angle equal to ADC, and join A with the point of intersection F; AF is the straight line required.

For inflect CN=CA, and join GF and AH.

The triangles AHC and AEC are equal; for the angle AFG, being by construction equal to ADC, is equal (I. 23. El.) to the alternate angle formed by the production of BA with AD, and consequently (III. 25. cor. El.) AB touches the circle at A; whence the angle BAH=HFA=DAE, and taking these from the equal angles BAC and DAC, there remains CAH=CAE, but the angles ACH and ACE are also equal, and the side AC is common to the two triangles; wherefore AH=AE, and CH=CE. And because the triangles ADE and AFG are similar, AD : AE : : AF : AG, and AD.AG=AE.AF. Again, the triangles

ANC and ACD being similar, AN : AC : : AC : AD, and AN.AD=AC^2. But FC : EF : : AD : AE : : AF : AG, and CE : EF : : AB or AD : AF; consequently FC.CE : EF^2 :: AD : AG : : NG.AD : NG.AG; and since AC bisects the angle FAH, FC.CH + AC^2 = FA.AH = FA.AE = AG.AD = AN.AD + NG.AD, it follows that FC.CH, or FC.CE= NG.AD, and hence EF^2 = NG.AG. Now K^2 = CG^2—AC^2= (II. 23. cor.) NG.AG; wherefore EF^2 = K^2, and EF=K*.

PROP. XXVI. PROB.

Through two given points, to describe a circle touching a straight line given in position.

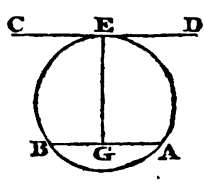
Let it be required to describe a circle through the points A, B, and touching the straight line CD.

It is evident that CD must either be parallel or inclined to the straight line which joins the points A and B.

1. *Let CD be parallel to AB.*

ANALYSIS.

From the point of contact E, draw (I. 6. El.) EG perpendicular to CD. Hence (III. 24. cor. El.) EG passes through the centre of the circle, and since it is also perpendicular to AB (I. 23. El.) it bisects that chord at right angles (III. 4. El.) the point G is therefore given, and the perpendicular GE; consequently the three points A, E, and B being thus given, the circle AEB is given.



COMPOSITION.

Draw (I. 7. El.) GE bisecting AB at right angles, and (III. 10. cor. El.) through the points, A, E and B describe a circle; this will touch the straight line CD.

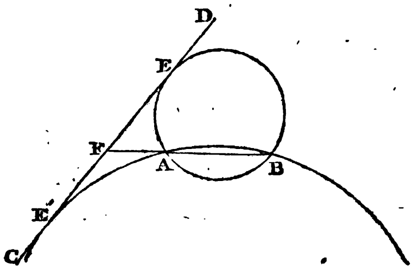
* See Note LVI.

For (III. 6. El.) GE must pass through the centre of the circle, and (I. 23. El.) it meets the parallels CD and AB at right angles; whence (III. 24. El.) CD is a tangent to the circle.

2. *Let CD be inclined to AB.*

ANALYSIS.

Produce BA to meet CD in F. Then (III. 32. cor. 2. El.) $FE^2 = AF.FB$; but the point of concourse being given, the rectangle AF, FB is given, and consequently FE and the point E. Wherefore since the three points A, E, and B are given, the circle AEB is given.



COMPOSITION.

Produce BA to meet CD in F, find (VI. 18. El.) FE or FE′ a mean proportional to AF and FB, and (III. 10. cor. El.) through the points A, B, and E, or A, B, and E′, describe a circle; this will touch the straight line CD.

For since AF : FE : : FE : FB, therefore (V. 6. El.) $FE^2 = AF.FB$, and consequently (III. 34. El.) FE, or FE′, touches the circle.

PROP. XXVII. PROB.

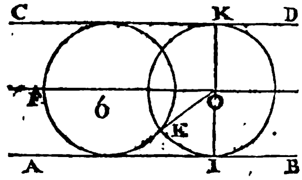
Through a given point, to describe a circle touching two straight lines given in position.

Let it be required, through the point E, to describe a circle touching AB and CD.

1. *Suppose AB parallel to CD.*

ANALYSIS.

Through the centre O draw the parallel FO and the common perpendicular KI. It is evident that the radius OI is given, and consequently FO is given in position; but OE, being equal to OI, is given, and therefore the centre O is given.



COMPOSITION.

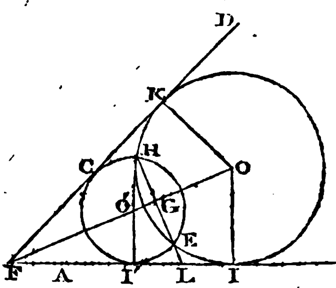
Draw a parallel FO bisecting the distance between the straight lines AB and CD, and from E with a radius equal to half that distance intersect FO in O, or O′; this point is the centre of the circle required. For $OE = OI = OK$, and the circle which passes through E must touch at K and I.

2. *Suppose CD inclined to AB.*

ANALYSIS.

Produce BA and DC to meet in F, join OI, OK, and OF, and from E draw EGH perpendicular to OF.

The triangles OKF and OIF, being (III. 24. El.) right-angled, and having the side OK equal to OI and the side OF common, are (I. 22. El.) equal, and consequently the angle OFK is equal to OFI; wherefore since the point of concourse F is given, the straight line OF is given. But, the point E being given, the perpendicular EH is thence given, and (III. 4. El.) GH being equal to GE, the opposite point H is given. Two points E, H, and a straight line AB, are thus given, and therefore, by the last proposition, the circle EHKI is given.



COMPOSITION.

Produce BA and DC to meet in F, draw (I. 5. El.) FO bisecting the angle BFD, from E (I. 6. El.) let fall the perpendicular EG, and extend it both ways, making GH=GE, find (VI. 18. El.) LI, or LI′, a mean proportional to HL and LE, and through the points H, E, I, or H, E, I′, describe a circle; this circle will touch both the straight lines AB and CD.

For the centre of the circle which passes through E and H, must (III. 5. El.) occur in FO; let it be O, join OI and draw the perpendicular OK. Because $HL.LE=LI^2$, the circle touches AB at I, and hence OIF is a right angle; consequently the triangles KOF and IOF having the angles OKF and OFK equal to OIF and OFI, and the side OF common, are (I. 21. El.) equal, and therefore OI=OK; whence the circle described from O passes through K, and (III. 24. El.) must touch CD at that point.

Cor. If the given point E should fall on AB, and thus coincide with the point of contact,—the problem will become much simpler; for the centre O, lying in the intermediate or bisecting line FO, will be determined by the intersection of the perpendicular IO.

PROP. XXVIII. PROB.

Through two given points, to describe a circle touching a given circle.

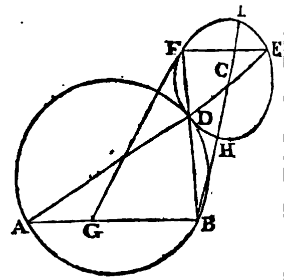
Let it be required, through the points A and B, to describe a circle, touching another circle whose centre is C.

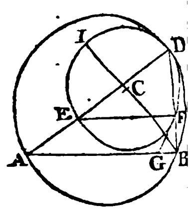
ANALYSIS.

Through D, the point of contact, draw ADE and BDF, join EF, at F (I. 5. cor. 2. El.) apply the tangent FG, and draw BHCI.

Because FG touches the given circle, the angle BFG is (III. 25. El.) equal to FED, and therefore equal to BAD,

since (III. 29. El.) FE and AB are parallel; but the triangles BGF and BDA have likewise a common angle at B, and are hence similar; wherefore BF : BG : : BA : BD, and (V. 6. El.) BA.BG = BF.BD = (III. 32. El.) BI.BH. But BI and BH are given, and thence the rectangle BA, BG is given, and consequently (II. 9. El.) the point G is given. Hence the tangent GF, and D, the intersection of BF, are given; wherefore the circle that passes through the three points A, D, and B, is given.





COMPOSITION.

Make (VI. 3. El.) BA BI : : BH : BG, draw (III. 26. El. the tangent GF, join BF cutting the given circumference in D, and (III. 10. cor. El.), through the points A, D, and B describe a circle; this will touch the circle FDE.

For draw ADE, join FE, and draw BHCI. Since BA : B : : BH : BG, therefore (V. 6. El.) BA.BG = BI.BH = (III. 32. El.) BF.BD; whence BF : BG : : BA : BD, an consequently the triangles BGF and BDA, having the sam vertical angle, are (VI. 14. El.) similar, and hence the ang BFG is equal to BAD. But (III. 25. El.) BFG is equal FED, and thus the alternate angles BAE and FEA a equal, and FE is parallel to AB; whence (III. 29. El.) t two circles touch at D.

PROP. XXIX. PROB.

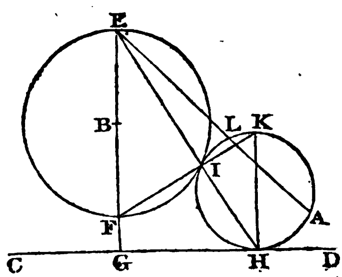
Through a given point, to describe a cir touching a given circle and a straight line whic given in position.

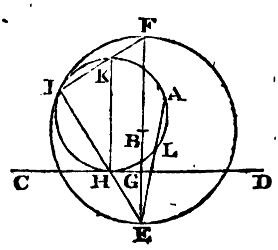
∴ Let it be required, through the point A, to describe a circle touching the straight line CD and the circle whose centre is B.

ANALYSIS.

From the centre of the given circle let fall the perpendicular EBG, join EI and extend it to H in the straight line CD, also draw FIK and join HK.

The angle HIK, being equal to EIF which stands in a semicircle, is (III. 22. El.) a right angle, and consequently HK is the diameter of the circle ILA, and H the point of contact. The triangles HEG and FEI are therefore similar, HE : EG :: EF : EI, whence HE.EI = EG.EF. Join ELA, and (III. 32. El.) AE.EL = HE.EI = EG.EF; but the rectangle EG, EF is given, and consequently that of HE, EI, and EH being given, the point L is hence given. Wherefore, since the two points A, L, and the straight line CD, are all given,—the circle HIA is given.





COMPOSITION.

Join EA, draw the perpendicular EG, make (VI. 3. El.) AE : EG :: EF : EL, and by Prop. 26. of this Book, describe a circle through the points A, L, and touching the straight line CD; this circle will also touch the given circle.

For draw the diameter HK, join EH cutting the circumference EIF, and draw FIK meeting HK.

The triangles HEG and FEI being evidently similar,

HE : EG :: EF : EI, and HE.EI = EG.EF; but AE : EG :: EF : EL, and AE.EL = EG.EF; wherefore HE.EI = AE.EL, and (III. 34. El.) the point I must lie in the circumference HIK. But the two circles also touch in I; for EG being parallel to HK, the angles IEF and IHK are equal, which are again equal to those made by a tangent with IF and IK.

Cor. The problem will be greatly simplified, if the given point A should occur in the straight line or in the circle, and hence coincide with either of the points of contact H or I; for EIH and FIK being drawn, the perpendicular HK is the diameter of the required circle.

PROP. XXX. PROB.

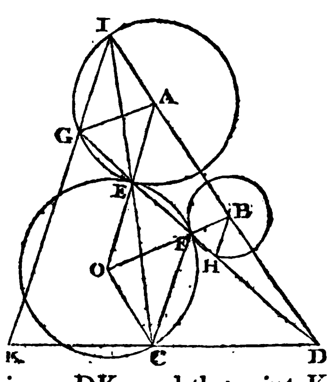
Through a given point, to describe a circle *touching* two given circles.

Let it be required, through the point C, to describe a circle touching two given circles whose centres are A and B.

ANALYSIS.

Join AB, and produce it to meet, in D, the extension of the straight line which connects E, F, the points of contact; join OA and OB, AG and BH, draw CEI, and produce IG and DC to meet in K.

The isosceles triangles EOF, EAG, and FBH, are evidently similar, and consequently AG is parallel to BF and AE to BH. Whence (VI. 2. El.) AE : BH :: AD : BD; and, this ratio being therefore given, the point D is given. Again, AG : BF :: DG : DF, and DG : DF :: DK : DC, for (III. 29. El.) IG is parallel to FC; consequently, DC being given, DK, and the point K,



are given.. Wherefore, by Proposition 17. of the first Book of Analysis, the straight line GE, included by the reflected lines KI and CI, and directed to the given point D, is given; hence AEO is given in position. Join OC, and the angle ECO, being equal (I. 11. El.) to CEO, is given; and consequently CO, and the centre O, are given.

COMPOSITION.

Make (VI. 3. El.) AE : BH : : AD : BD, join DC, make BH : AE : : DC : DK; and, from the points K and C, inflect KI and CI, by Prop. 17. Book I. such that GE shall tend to D, produce AE and CO, making the angle ECO equal to CEO; the intersection O is the centre of the required circle.

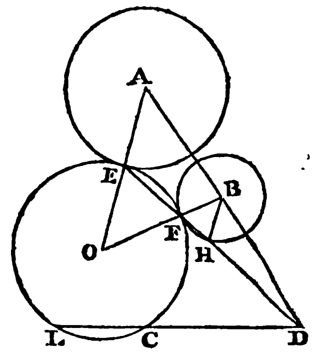
For join AG, CF, OB, and BH. Because AE or AG : BH or BF : : AD : BD, and the triangles ADG and BDF have a common angle at D, they are (VI. 15. El.) similar; consequently AD : BD : : DG : DF : : DK : DC, and IG is parallel to FC; and therefore the circles touch at E. But the triangle, BFH, having its sides BF and BH parallel to AG and AE, the sides of the isosceles triangle GAE, must likewise be isosceles; wherefore the circles meet at F: And, since BH is parallel to EO, they must touch at that point. Again, the angle ECO being equal to CEO, the side OE is equal to OC; and consequently the circle described from O, and which touches at E and F, must also pass through C.

Otherwise thus.

ANALYSIS.

Join the centres A, B and O, and produce AB and the straight line connecting E, F, the points of contact, till they meet in D; join also BH and DC, and extend this to cut the circle in L.

Since EOF and FBH are isosceles triangles, the vertical angles OFE and BFH are equal to OEF and BHF, which are therefore equal, and consequently (I. 23. El.) EO is parallel to BH; whence (VI. 2. El.) AE : BH : : DA : DB, and the point D is therefore given. Again (VI. 1. El.) DA : DB : : DE : DH, or (V. 25. cor. 2. El.) DE.DF : DH.DF; but the



rectangle DH, DF, being equal (III. 32. El.) to the rectangle under the segments of DB intercepted by the circle B, is given, and hence DE.DF or DC.DL are given rectangles; wherefore DC being given, DL and the point L are likewise given. The problem is thus reduced to Proposition 28. of this Book.

COMPOSITION.

Make (VI. 3. El.) AE : BH : : AD : BD, join DC, and produce it to L, such that the rectangle DC, DL shall be to the rectangle formed by a secant drawn from D to the circle B, in the ratio of AE to BH, and (II. 28. Anal.) describe a circle through the points C and L, and touching the circle A; this will also touch the circle B.

For join OA, OB, EH, and draw BH parallel to AO. Because AE : BH : : AD : BD, it is evident that EH, being produced, will meet AD in D; hence AE : BH : : DE : DH, or (V. 25. cor. 2. El.) DE.DF : : DH.DF; but, by construction, AE : BH : : DC.DL : DH.DF, and consequently DC, DL is equal to DE, DF, and the point F lies in the circle O. Wherefore the triangle EOF is isosceles, and likewise the similar triangle HBF; hence F belongs also to the circle B and (III. 28. El.) is the point of mutual contact.

If L should coincide with the point C, the construction will be effected by the corollary to the preceding Proposition *.

PROP. XXXI. PROB.

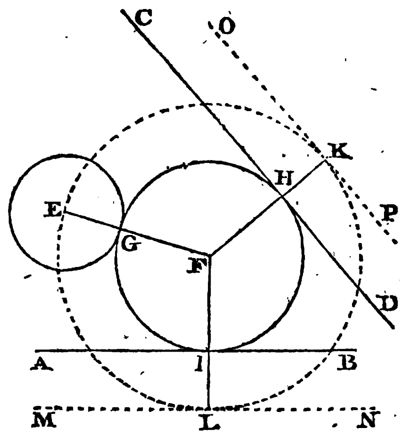
To describe a circle that shall touch a given circle and two straight lines given in position.

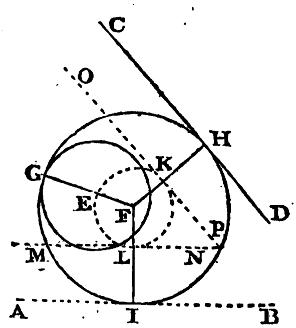
Let it required to describe a circle touching the straight lines AB and CD, and another circle whose centre is E.

ANALYSIS.

Join FE, draw FH,FI to the points of contact, from F, with the radius FE, describe a circle meeting FH and FI produced in K and L, and, at these points, apply the tangents MN and OP.

Because FE=FK=FL and FG=FH=FI, therefore GE=HK=IL. But the tangents CD and OP, being perpendicular to FK, are parallel; and, for the same reason, the tangents AB and MN are parallel. Wherefore OP and MN are given in position, and consequently, by Prop. 27. the circle EKL is given; and thence the concentric circle GHI.





* See Note LVII.

COMPOSITION.

At a distance equal to the radius of the given circle, draw MN and OP parallel to AB and CD; and, by Prop. 27. of this Book, find F the centre of a circle which passes through E and touches MN and OP; F is likewise the centre of the required circle.

For join FE, and draw FK and FL to the points of contact. And because GE=HK=IL, it is evident that FG=FH=FI. But the circle also touches at the points H and I, since CD and AB are perpendicular to FK and FL.

Scholium. The six preceding propositions are only cases of a general problem: " Three things being given,—whether points, or straight lines, or circles,—to describe a circle limited by them all." This problem comprizes ten distinct cases. Two of these have been already given in the Elements: To describe a circle through three given points, forms the 10th Prop. Book III.: To describe a circle that shall touch three straight lines given in position, is the basis of Prop. 10. Book IV., and appears complete in the construction of Prop. 31. Book VI. The same principle, it may be perceived, runs through all the solutions already given; the conditions of the problem are only repeatedly simplified, each of the linear or circular data being exchanged in succession for a point. Two cases still remain: When there are given three circles or two circles and a straight line, to describe another circle limited by these data. These are easily reduced, however, to the cases already solved, as in the concluding proposition,—by drawing a parallel, or describing a concentric circle, at distances, according to the relative position of the data, equal to the sum or difference of the given radii *.

---

* See Note LVIII.

GEOMETRICAL ANALYSIS.

BOOK III.

DEFINITION.

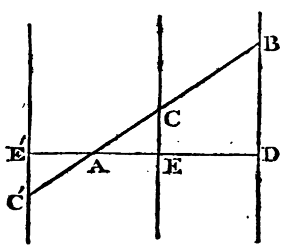
If a point vary its position according to some determined law, it will trace a line which is termed its *Locus*.

PROP. I. THEOR.

If a straight line, drawn through a given point to a straight line given in position, be divided in a given ratio, the *locus* of the point of section is a straight line given in position.

Let the point A and the straight line BD be given in position, and let AB, limited by these, be cut in a given ratio at C; this point will lie in a straight line which is given in position.

ANALYSIS.



From A let fall the perpendicular AD upon BD, and, through C, draw CE parallel to BD. It is evident (VI. 1. El.) that AC : AB :: AE : AD, and consequently that the ratio of AE to AD is given; but AD is given both in position and magnitude, and hence AE and the point E are given, and therefore CE, which stands at right angles to AD, is given in position.

COMPOSITION.

Let fall the perpendicular AD, which divide at E in the given ratio, and erect the perpendicular CE; this straight line is the *locus* required. For CE being parallel to BD, AC : AB : : AE : AD, that is, in the given ratio.

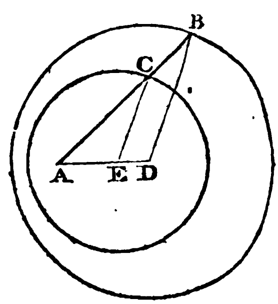
PROP. II. THEOR.

If a straight line, drawn through a given point to the circumference of a given circle, be divided in a given ratio, the *locus* of the point of section will also be the circumference of a given circle.

Let AB, terminating in a given circumference, be cut in a given ratio; the segment AC will likewise terminate in a given circumference.

ANALYSIS.

Join A with D the centre of the given circle, and draw CE parallel to BD. It is obvious (VI. 1. El.) that AC : AB : : AE : AD; whence the ratio of AE to AD being given, AE and the point E are given. Again, since (VI. 2. El.) AC : AB : : CE : BD, the ratio of CE to BD is given, and consequently CE is given in magnitude. Wherefore the one extremity E being given, the other extremity of CE must trace the circumference of a given circle.



COMPOSITION.

Join AD, and divide it at E in the given ratio, and in the same ratio make DB to the radius EC, with which, and from the centre E, describe a circle.

GEOMETRICAL ANALYSIS.

BOOK III.

DEFINITION.

If a point vary its position according to some determined law, it will trace a line which is termed its *Locus*.

PROP. I. THEOR.

If a straight line, drawn through a given point to a straight line given in position, be divided in a given ratio, the *locus* of the point of section is a straight line given in position.

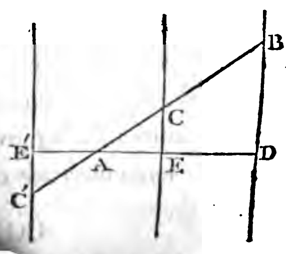
Let the point A and the straight line BD be given in position, and let AB, limited by these, be cut in a given ratio at C; this point will lie in a straight line which is given in position.

ANALYSIS.

From A let fall the perpendicular AD upon BD, and, through C, draw CE parallel to BD. It is evident (VI. 1. El.) th... B

: AF ... and

that ... F



... and the point E are

... right angles to AD,

the angle CEA is equal to BDA, and therefore a right angle; consequently the straight line EC is given in position.

COMPOSITION.

Having let fall the perpendicular AD, and made the angle DAE equal to BAC, make AD to AE in the given ratio, and, at right angles to AE, draw EC; this is the *locus* required. For the triangles BAD and CAE, having their vertical angles equal, and the angles at D and E right angles, are similar, and consequently AB : AD : : AC : AE, or alternately AB : AC : : AD : AE, that is, in the given ratio.

PROP. IV. THEOR.

If, through a given point, two straight lines be drawn in a given ratio, and containing a given angle; if the one terminate in a given circumference, the other will also terminate in a given circumference.

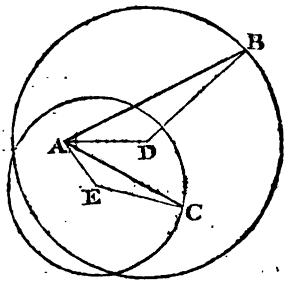
Let the angle BAC, its vertex A, and the ratio of its sides, be given; if AB be limited by a given circle, the *locus* of C will also be a given circle.

ANALYSIS.

Join A with D the centre of the given circle, draw AE at the given angle with AD, and in the given ratio, and join DB and EC.

Because the point A and the centre D are given, the straight line AD is given; and since the angle DAE, being equal to BAC, is given, AE is given in position. But AD being to AE in the given ratio, AE must be given also in

magnitude, and consequently the point E is given.



Again, the whole angle BAC being equal to DAE, the part BAD is equal to CAE; and because AB : AC : : AD : AE, alternately AB : AD : : AC : AE; wherefore the triangles ADB and AEC are similar, and hence AB : BD : : AC : CE, or alternately AB : AC : : BD : CE; consequently the fourth term CE is given in magnitude; and its extremity E being given, the other must lie in a given circumference.

COMPOSITION.

Having drawn AE at the given angle with AD, make AD to AE in the given ratio, and in the same ratio let DB be made to EC; a circle described from the centre E with the distance EC, is the *locus* required.

For AD : AE : : DB : EC, and alternately AD : DB : : AE : EC; but the angle BAD is equal to CAE, because the whole BAC is equal to DAE; consequently the triangles ABD and ACE are similar, and AB : AD : : AC : AE or alternately AB : AC : : AD : AE, that is, in the given ratio.

Scholium. Since the tangent of a circle is only the extreme limit of its adjacent arc, which, in proportion as the circle expands, must continually approach to that ultimate position—the rectilineal, may be derived from the circular, *locus*. Thus, in Prop. 2. of this Book, if the centres E and D be supposed to retire to a distance indefinitely remote, the arcs which pass through C and B may be viewed as merging in their tangents or in perpendiculars let fall from those points upon AD, which is the first proposition. In like manner, if the circles in Prop. 4. be supposed immeasurably expanded, the arcs in which the points B and C lie may be conceived to pass into tangents perpendicular to AD and AE, as in Prop. 3.

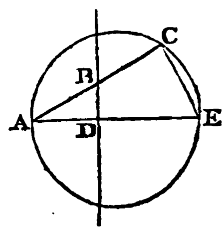
PROP. V. THEOR.

If a straight line, drawn from a given point to a straight line given in position, contain a given rectangle, the *locus* of its point of section will be a given circle.

Let the rectangle AB, AC be given, while the point B and the straight line BD are given in position; the point C will lie in the circumference of a given circle.

ANALYSIS.

Draw AD perpendicular to BD, and make the rectangle AD.AE=AB.AC. Since AD is evidently given both in position and magnitude, AE and the point E are given. Join CE. Because AD.AE=AB.AC, AD : AB : : AC : AE, and the triangles DAB and CAE, having the sides about the common angle at A proportional, are therefore similar; and consequently the angle ACE is equal to ADB, or a right angle. Whence (III. 22. El.) the point C must lie in a semicircle, of which AE the diameter is given.



COMPOSITION.

Having drawn the perpendicular AD, make the rectangle AD, AE equal to the given space, and upon the diameter AE describe a circle; this is the *locus* required. For draw AC and CE. The triangles ABD and AEC are similar, since they have a common angle at A, and those at D and C right angles; wherefore AB : AD : : AE : AC, and AB.AC= AD.AE, that is, equal to the given space.

PROP. VI. THEOR.

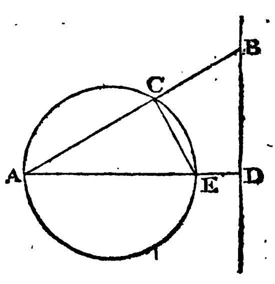
If a straight line, containing a given rectangle, be drawn through a given point to the circumference of a given circle, the *locus* of its point of section will be either a straight line given in position or a given circle, according as it originates or not in the given circumference.

Let the rectangle AC, AB be equal to a given space, and the segment AC terminate in a given circumference, the point of origin A may lie either in that circumference or not.

1. *Suppose the given point A lies in the given circumference; the* locus *of B is a straight line given in position.*

ANALYSIS.

Draw the diameter AE, and make AE.AD = AB.AC; wherefore the point D is given. Join CE and BD; and because AE.AD =AB.AC, AC : AE : : AD : AB; whence the triangles CAE and DAB, having likewise a common angle at A, are similar. Consequently the angle ADB being thus equal to ACE, is a right angle, and the straight line DB is hence given in position.



COMPOSITION.

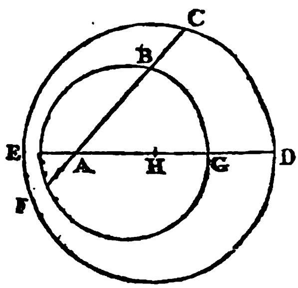
Having drawn the diameter AE, make the rectangle AE,AD equal to the given space, and erect the perpendicular DB; this is the *locus* required. For draw ACB, and join CE. The right-angled triangles ACE and ADB being evidently

similar, AC : AE : : AD : AB, and AC.AB=AE.AD, or the given space.

2. *Suppose that the point A does not lie in the given circumference; then the locus of B is a given circle.*

ANALYSIS.

Draw the diameter EAD, and produce CAF to the circumference. The rectangle AC, AF, being equal to AD, AE, is given, and has therefore a given ratio to the rectangle AC, AB; whence the ratio of AF to AB is given, and consequently (III. 2. Anal.) AB terminates in the circumference of a given circle.



COMPOSITION.

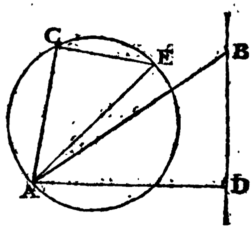
Having drawn the diameter EAD, make the rectangle AD, AH equal to the given space, and (III. 2. Anal.) describe a circle EBGF, such that a straight line, passing through it shall be cut by the circumference in the ratio of AE to AH; this circle is the *locus* required. For AE : AH : : AF : AB : : AF.AC : AB.AC; wherefore AF.AC : AB.AC : : AE.AD : AH.AD, and the first term of this analogy being equal to the third, the second term is equal to the fourth, or AB.AC=AH.AD, that is, equal to the given space.

PROP. VII. THEOR.

If two straight lines, containing a given rectangle, be drawn from a given point at a given angle; should the one terminate in a straight line given in position, the other will terminate in the circumference of a given circle.

Let the point A, the angle BAC, and the rectangle under its sides BA, AC be given; if the direction BD be given, then will the *locus* of C be a given circle.

ANALYSIS.



From A let fall the perpendicular AD upon BD. Draw AE, to contain with AD an angle equal to the given angle, and a rectangle equal to the given space; and join CE.

Since AD is evidently given in position and magnitude, AE is likewise given in position and magnitude; and the rectangle AD, AE being equal to AB,AC, therefore AD : AB : : AC : AE; but the angle DAE is equal to BAC, and hence DAB is equal to EAC. Wherefore the triangles ABD and AEC, having each an equal angle and its containing sides proportional, are similar; and consequently the angle ACE is equal to the right angle ADB. Whence the *locus* of C is a circle, with AE for its diameter.

COMPOSITION.

Having let fall the perpendicular AD, draw AE, making the angle DAE equal to the given angle, and the rectangle DA, AE equal to the given space, and on AE, as a diameter, describe a circle; this is the *locus* required.

For join CE; and the triangles DAB and EAC being right-angled at D and C, and having the vertical angles at A equal, are evidently similar, and consequently AD : AB : : AC : AE; and hence the rectangle AB, AC is equal to AD, AE, that is, to the given space.

COMPOSITION.

At a distance equal to the radius of the given circle, draw MN and OP parallel to AB and CD; and, by Prop. 27. of this Book, find F the centre of a circle which passes through E and touches MN and OP; F is likewise the centre of the required circle.

For join FE, and draw FK and FL to the points of contact. And because GE=HK=IL, it is evident that FG=FH=FI. But the circle also touches at the points H and I, since CD and AB are perpendicular to FK and FL.

Scholium. The six preceding propositions are only cases of a general problem: "Three things being given,—whether points, or straight lines, or circles,—to describe a circle limited by them all." This problem comprizes ten distinct cases. Two of these have been already given in the Elements: To describe a circle through three given points, forms the 10th Prop. Book III.: To describe a circle that shall touch three straight lines given in position, is the basis of Prop. 10. Book IV., and appears complete in the construction of Prop. 31. Book VI. The same principle, it may be perceived, runs through all the solutions already given; the conditions of the problem are only repeatedly simplified, each of the linear or circular data being exchanged in succession for a point. Two cases still remain: When there are given three circles or two circles and a straight line, to describe another circle limited by these data. These are easily reduced, however, to the cases already solved, as in the concluding proposition,—by drawing a parallel, or describing a concentric circle, at distances, according to the relative position of the data, equal to the sum or difference of the given radii *.

* See Note LVIII.

GEOMETRICAL ANALYSIS.

BOOK III.

DEFINITION.

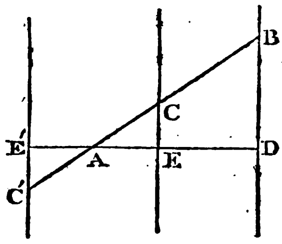
If a point vary its position according to some determined law, it will trace a line which is termed its *Locus*.

PROP. I. THEOR.

If a straight line, drawn through a given point to a straight line given in position, be divided in a given ratio, the *locus* of the point of section is a straight line given in position.

Let the point A and the straight line BD be given in position, and let AB, limited by these, be cut in a given ratio at C; this point will lie in a straight line which is given in position.

ANALYSIS.



From A let fall the perpendicular AD upon BD, and, through C, draw CE parallel to BD. It is evident (VI. 1. El.) that AC : AB : : AE : AD, and consequently that the ratio of AE to AD is given; but AD is given both in position and magnitude, and hence AE and the point E are given, and therefore CE, which stands at right angles to AD, is given in position.

the contained angle ABE is given, the triangle BEA is likewise given in species; and thence the point A, and the straight line EA, are given in position.

COMPOSITION.

Having assumed in EH any point H, draw HGF in the given inclination, make FG : FH : : NM : MO, and produce HF till KN : OL : : FG : IF; EI is the straight line required. For BC : AB :: FG : IF :: KN : OL, and AB.KN = BC.OL; but BC : BD : : FG : FH : : NM : MO, and BC.MO = BD.NM. Wherefore AB.KN = BC.OL = BC.ML + BD.NM, and AB.KM = AB.NM + BC.ML + BD.NM = BC.ML + AD.NM, and hence AB.KL = AB.ML + BC.ML + AD.NM = AC.ML + AD.NM.

PROP. X. THEOR.

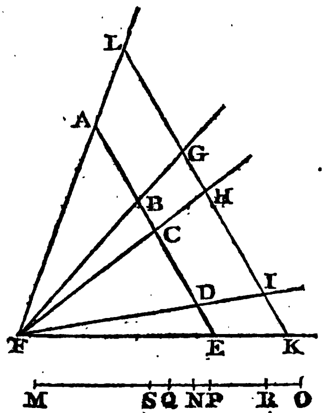
Four diverging lines being given in position, if a straight line cut them at given angles, and such that the rectangles of its first and second segments by given lines shall be equal to both the rectangles of its third and fourth segments by given lines; the *locus* of its point of origin will be a straight line given in position.

Let ABCDE cut the diverging lines FG, FH, FI, and FK at given angles, and let AB.MN + AC.NO = AD.OP + AE.PQ; then will the *locus* of the point A be a straight line given in position.

ANALYSIS.

Because AB.MN + AC.NO = AD.OP + AE.PQ, it follows, by decomposition, that AB.MO + BC.NO = AB.OQ +

BD.OP + BE.PQ, and consequently AB.MQ + BC.NO = BD.OP + BE.PQ. Make BD : BC : : NO : OR, and BD : BE : : PQ : PS; then BD.OR = BC.NO, and BD.PS = BE.PQ; whence AB.MQ + BD.OR = BD.OP + BD.PS, or AB.MQ = BD.SR, and, therefore, AB : BD :: SR : MQ. But the triangle BDF being given in species, the ratio of BD to BF is given; and consequently the ratio of AB to BF is given, and the contained angle ABF being given, the triangle BFA is likewise given in species; and hence the straight line FA is given in position.



COMPOSITION.

Having assumed in FK any point K, draw KIHG at the given inclination, make GI : GH : : NO : OR, and GI : GK : : PQ : PS, and produce KG till MQ : SR : : GI : GL; FL is the straight line required.

For BD : BC : : GI : GH : : NO : OR, and BD.OR = BC.NO; but BD : BE : : GI : GK : : PQ : PS, and BD.PS = BE.PQ; again, MQ : SR : : GI : GL : : BD : AB, and AB.MQ = BD.SR. Whence AB.MQ + BC.NO = BD.SR + BD.OR = BD.SO = BD.PS + BD.OP = BE.PQ + BD.OP, add to each AB.OQ = AB.NQ + AB.NO, or AB.PQ + AB.OP, and AB.MN + AC.NO = AD.OP + AE.PQ *.

PROP. XI. THEOR.

If a straight line given in position, be cut at given angles by two straight lines, which intercept, from

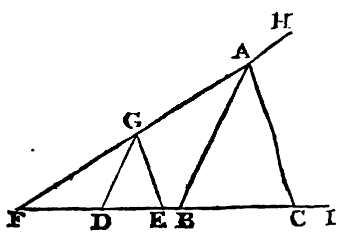
* See Note LVIII.

two given points in it, segments that have a given ratio, the *locus* of the point of concourse is a straight line given in position.

Let AB and AC be drawn, such that the angles ABF, and ACF, with the ratio of DB to EC, are given; the *locus* of A, the point of concourse, is a straight line given in position.

ANALYSIS.

Make FD to FE in the given ratio, and join FA. Since therefore FD : FE : : DB : EC, it follows (V. 19. El.) that FD : FE : : FB : FC; consequently the ratio of FB to FC, and thence that of FB to BC, are each given. But the angles FBA and FCA being given, the triangle BAC is evidently given in species, and therefore the ratio of AB to BC is given, and hence the ratio of FB to AB is also given. The triangle FBA having thus two sides containing a given angle and in a given ratio, is (VI. 14. El.)



given in species; and consequently the angle BFA is given, and the straight line FA given in position.

COMPOSITION.

Having made FD to FE in the given ratio, draw DG and EG at the given angles with FI, and join F with their point of concourse; FGH is the *locus* required.

For, from any point A in FH, draw AB and AC at the given angles with FI, and consequently parallel to GD and GE. Because AB is parallel to GD, and AC to GE, FG : FA : : FD : FB : : FE : FC (VI. 1. El.) and alternately FD : FE : : FB : FC; wherefore (V. 19. cor. 1. El.) DB : EC : : FD : FE, that is, in the given ratio.

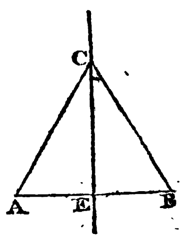
PROP. XII. THEOR.

If, from two given points, there be inflected two straight lines in a given ratio, the *locus* of their point of concourse is a straight line, or a circle given in position.

Let AC and BC, drawn from the points A and B, have a given ratio; then will C, the point of concourse, lie in a straight line given in position, or in the circumference of a given circle.

1. *When the inflected lines are equal, they terminate in a straight line given in position.*

ANALYSIS.



Bisect AB in E, and join EC. The triangles ACE and BCE, having the sides AE and AC equal to BE and BC, and EC common, are equal (I. 2. El.); wherefore the angle AEC is equal to BEC, and EC is perpendicular to AB, and consequently given in position.

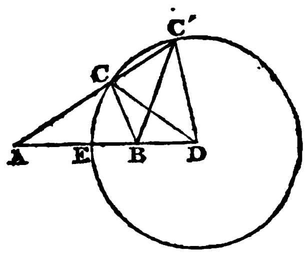
COMPOSITION.

Bisect AB by the perpendicular EC, which is the *locus* required. For draw AC and BC to any point in it, and the triangles AEC and BEC are (I. 3. El.) evidently equal, and hence AC is equal to BC.

2. *When the inflected lines AC and BC have an unequal ratio, their point of concourse lies in the circumference of a given circle.*

ANALYSIS.

Draw CD, making the angle BCD equal to BAC, and meeting AB produced in D. The triangles DAC and DCB, having the angle at D common, and the angles at A and C equal, are evidently similar; and hence AD : AC :: DC : BC,



and alternately AD : DC : : AC : BC, that is, in the given ratio. But AD : DC :: DC : BD, and consequently AD is to BD in the duplicate of the given ratio of AD to DC, and which is therefore likewise given. Consequently BD, and the point D, are given; and DC being thence given, its extremity C must lie in the circumference of a circle described with that radius.

COMPOSITION.

Divide AB in the given ratio at E, and in the same ratio make ED to BD; the circle described from the centre D, and with the radius DE, is the *locus* required.

For, since AE : BE : : ED : BD, it follows (V. 19. El.) that AD : ED, or DC : : ED, or DC : BD; hence the triangles DAC and DCB, thus having the sides which contain their common angle at D proportional, are similar, and therefore AC : AD : : BC : DC, or alternately AC : BC : : AD : DC or ED, that is, in the given ratio.

Scholium. Since, in the second case, AC : BC :: AD : ED, it is obvious, that as the ratio of AC to BC approaches to equality, the centre D must continually recede from A or E, and consequently the arc EC may be conceived as ultimately passing into the tangent which bisects AB at right angles; thus comprehending the first case of the proposition.

PROP. XIII. THEOR.

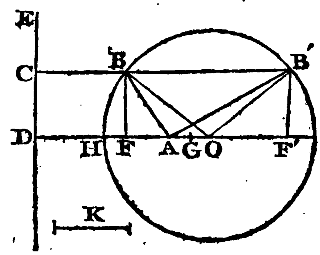
A point and a straight line being given in position, the *locus* of another point, the square of whose distance from the former, is equal to the rectangle under its distance from the latter and a given straight line—is a given circle.

The point A and the straight line DC being given in position, let the square of BA be equal to the rectangle under the perpendicular BC and K; the *locus* of B is a given circle.

ANALYSIS.

Draw DFA parallel to CB, make AO equal to the half of K, and bisect it in G, join BO, and let fall the perpendicular BF.

Because AO is bisected in G, $OB^2—AB^2$, or $AB'^2—OB'^2$, (II. 24. El.) $= 2AO.GF = K.GF$; but $AB^2 = K.BC$, or K.DF, and hence $OB^2 = K.DG$. Since, therefore DG is given, OB is also given; and the one extremity O being given, the other extremity B must lie in the circumference of a given circle.



COMPOSITION.

Having drawn DA parallel to CB, make $AO = \frac{1}{2}K$, and $AG = \frac{1}{2}AO$, and find OH a mean proportional between K and DG; a circle described from O with the radius OH, is the *locus* required.

For $OB^2—AB^2$, or $AB'^2—OB'^2$, $= 2AO.GF = K.GF$; and since, by construction, OH^2, or OB^2, $= K.DG$, it follows that $AB^2 = K.DF$, or K.BC.

Cor. If the given point A lies in DC, or coincides with D, then $DG = \frac{1}{4}K$ and $OH = \frac{1}{2}K$, or the circle likewise passes through D; whence AB becomes a chord, and its square (VI. 16. cor. 1. El.) is equivalent to the rectangle under the segment DF, and the diameter or K.

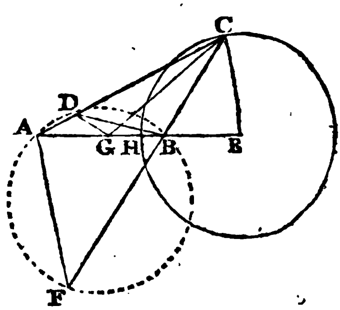
PROP. XIV. THEOR.

If, from two given points, there be inflected two straight lines, such that the difference of the square of the one and a given space, shall have to the square of the other, a given unequal ratio—their point of concourse will lie in the circumference of a given circle.

Let AC and BC be the inflected lines, and the rectangle AC, AD be made equal to the given space; then if the difference between the square of AC and that rectangle, or the remaining rectangle AC, CD, have a given unequal ratio to the square of BC, the *locus* of the point C will be a given circle.

ANALYSIS.

Make (VI. 4. El.) AE to BE in the given ratio, join CE and BD, produce CB to meet the circumference of a circle described about the triangle ADB, and join AF.



Because (III. 32. El.) the rectangle AC, CD is equal to FC, BC, it follows that the rectangle FC, BC is to the square of BC, or (V. 25. cor. 2. El.) FC is to BC, in the given ratio of AE to BE; wherefore (VI. 1. cor. 1. El.) AF is parallel to CE, and consequently the angle ECB is equal to AFB, which is equal to CDB the opposite exterior angle of the quadrilateral figure ADBF. Through the points C, D, B, describe a circle cutting AB in G, and join CG and DG; then (III. 32. El.) the rectangle BA, AG is equal to CA, AD, or to the given space, and hence AG, and the point G are given. The angle CDB, or ECB, is, therefore, equal to CGB, and consequently the triangles BEC and CEG are similar, and GE : CE : : CE :

BE; whence $CE^2 = GE.BE$, which is a given rectangle, and thus CE is given, and the *locus* of C a given circle.

COMPOSITION.

Make the rectangle AB, AG equal to the given space, and AE to BE in the given ratio, and find EH a mean proportional between GE and BE; the *locus* required is a circle described from E with the radius EH.

For, through the points A, D, B, and through C, B, G, describe circles, produce CB to F, and join AF, CG, and DG. Because $GE.BE = HE^2$, GE : HE or CE : : HE or CE : BE, and, therefore, the triangles GEC and CEB are similar, and the angle EGC is equal to ECB; but the angle EGC, or BGC, is equal to CDB, which again is equal to AFB; consequently the alternate angles ECB and AFB are equal, and the straight lines CE and AF parallel. Wherefore AE : BE : : FC : BC : : FC.BC, or AC.CD : BC^2. But CA.AD = BA.AG, or the given space; and hence the difference between the square of AC and that space, or the rectangle AC, CD, is to the square of BC, in the given ratio.

Scholium. If this local theorem were extended to the extreme cases, it would include other propositions which are exhibited in a separate form. Thus, supposing the given ratio to be that of equality, the sum or difference of the squares of AC and BC will be equivalent to the given space, according as this is greater or less than the square of AC. When the given space exceeds the square of AC, the centre E of the circle bisects AB, as in the first case of the sixteenth proposition of this Book. But when the square of AC is deficient by the given space, the ratio of AE to BE being that of equality, the centre E, lying beyond B, must be thrown to an infinite distance, and consequently the arc which crosses AB will merge in a tangent bisecting GB at right angles, as in Proposition 15. Again, if the deficient space be supposed to vanish, while the ratio of the squares of AC and BC, or that of

the inflected lines themselves is given, the point G will coincide with A, and the centre and radius of the circle are hence determined, after the same manner as in Proposition 12.

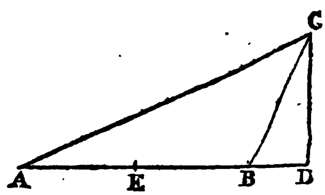
PROP. XV. THEOR.

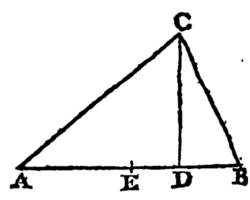
If from two given points there be inflected two straight lines, of whose squares the difference is given, the *locus* of their point of concourse will be a straight line given in position.

Let AC and BC, drawn from the points A and B, have the difference of their squares given; the *locus* of C, the point of concourse, is a straight line given in position.

ANALYSIS.

Draw CD perpendicular to AB, which bisect in E. The difference between the squares of AC and BC is (II. 24. El.) equal to twice the rectangle under AB and ED; consequently that rectangle, and its containing side ED, are given; whence the point of bisection E being given, the point D is given, and the perpendicular CD is therefore given in position.





COMPOSITION.

Bisect AB in E, and make (II. 9. El.) the rectangle under twice AB and ED equal to the given space; the perpendicular DC is the *locus* required.

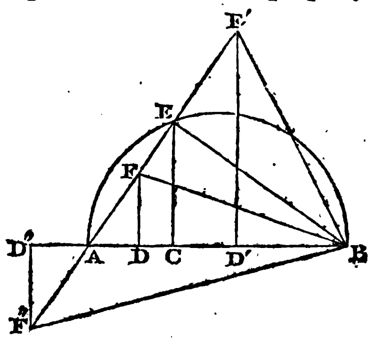
For (II. 24. El.) $AC^2-BC^2=AB.2ED=2AB.ED$, and consequently the difference of the squares of AC and BC is equal to the given space.

LEMMA.

If a straight line AB be cut any how in the point D, but divided at C, so that the segment AC shall be the n^{th} part of BC; then $n.AD^2+BD^2=AB.BC+(n+1)CD^2$.

For upon AB describe a semicircle, and erect the perpendicular CE, join AE, BE, draw DF parallel to CE and meeting AE or its extension, and join BF.

The angle AEB in a semicircle being a right angle, AC : CE : : CE : BC (VI. 16. cor. El.) and consequently (V. 24. El.) AC : BC : : $AC^2 : CE^2$; but $BC=n.AC$, and therefore $CE^2=n.AC^2$. Again, from the same property, AB : AE : : AE : AC, and AB : AC : : $AE^2 : AC^2$; and since $AB=(n+1)AC$; it follows (V. 5. El.) that $AE^2=(n+1)AC^2$. Now CE and DF being parallel, CE : DF : : AC : AD, and (V. 22. cor. 1. El.) $CE^2 : DF^2 : : AC^2 : AD^2$,



and CE^2 being equal to $n.AC^2$, therefore (V. 8. and 5. El.) $DF^2=n.AD^2$. In the same manner, it is shown that $EF^2=(n+1)CD^2$. But (VI. 16. cor. 1. El.) $BE^2=AB.BC$, and the triangles BDF and BEF being right angled, $BD^2+DF^2=BF^2=BE^2+EF^2$, and consequently by substitution, $n.AD^2+BD^2=AB.BC+(n+1)CD^2$.

PROP. XVI. THEOR.

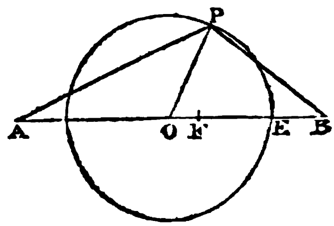
If, from given points, there be inflected straight lines, whose squares are together equal to a given space,—their point of concourse will terminate in the circumference of a given circle.

1. *When there are only two given points.*

Let AP and BP, drawn from the points A and B, have the sum of their squares given; the *locus* of their point of concourse is a given circle.

ANALYSIS.

Bisect AB in O, and join OP. The squares of AP and BP are (II. 25. El.) equal to twice the squares of AO and OP. Hence the sum of the squares of AO and OP is given; but AO and its square being given, the square of OP and OP itself must be given;



wherefore the *locus* of the extremity P is a circle, of which the point of bisection is the centre.

COMPOSITION.

Bisect AB in O, find (III. 33. El.) AF the side of a square equal to half the given space, and make (II. 14. El.) $OE^2 = AF^2 - AO^2$; the point O is the centre, and OE the radius, of the required circle.

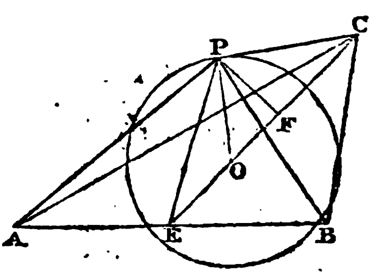
For (II. 25. El.) $AP^2 + BP^2 = 2AO^2 + 2OP^2 = 2AO^2 + 2OE^2 = 2AF^2$, or the given space.

2. *When three points are given.*

Let the straight lines AP, BP and CP, inflected from the points A, B, and C, have the sum of their squares given; the *locus* of their point of concourse is a given circle.

ANALYSIS.

Bisect AB in E, and (II. 25. El.) $AP^2 + BP^2 = 2AE^2 + 2EP^2$; consequently $AP^2 + BP^2 + CP^2 = 2AE^2 + 2EP^2 + CP^2$. Now $2AE^2 = AB.BE$, and, letting fall the perpendicular PF^2, (II. 11. El.) $2EP^2 = 2EF^2 + 2PF^2$, and $CP^2 = PF^2 + CF^2$. Wherefore $AP^2 +$



$BP + CP^2 = AB.BE + 3PF^2 + 2EF^2 + CF^2$. Trisect EC (I. 38. El.) in the point O, and join PO; and, by the *Lemma*, $2EF^2 + CF^2 = EC.CO + 3OF^2$. Whence $AP^2 + BP^2 + CP^2 = AB.BE + EC.CO + 3PF^2 + 3OF^2 = AB.BE + EC.CO + 3PO^2$. But the intermediate points of division E and O, are evidently given, and thence the rectangles AB, BE and EC, CO, are given; wherefore $3PO^2$ is given, and consequently PO itself. Since one extremity of that line then is given, the other extremity P must lie in the circumference of a given circle.

COMPOSITION.

Bisect AB in E, trisect EC in O, and find (III. 33. El.) OP such that its square shall be triple the excess of the given space above the rectangles AB, BE and EC, CO; the *locus* required is a circle, of which O is the centre, and PO the radius. For $3PO^2 = 3PF^2 + 3OF^2$, $3PO^2 + EC.CO = 3PF^2 + EC.CO + 3OF^2 = 3PF^2 + 2EF^2 + CF^2 = 2PE^2 + PF^2 + CF^2 = 2PE^2 + CP^2$; consequently the given space, or $3PO^2 + AB.BE + EC.CO = 2AE^2 + 2PE^2 + CP^2 = AP^2 + BP^2 + CP^2$.

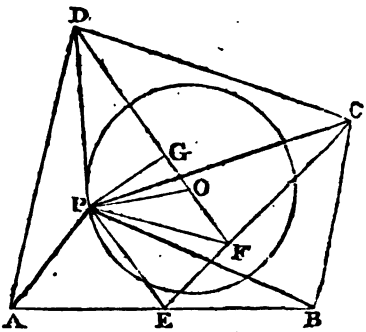
3. *When there are four given points.*

Let AP, BP, CP and DP drawn from the points A, B, C, and D, have the sum of their squares given; the *locus* of their concourse P is a given circle.

ANALYSIS.

Bisect AB in E, trisect EC in F, and join PE and PF. It is manifest, from the last case, that $AP^2 + BP^2 + CP^2 =$

$AB.BE + EC.CF + 3PF^2$; add DP^2 to each, and $AP^2+BP^2+CP^2+DP^2 = AB.BE+EC.CF+3PF^2+DP^2$. Let fall the perpendicular PG upon DF, and the given space is equal to $AB.BE + EC.CF + 3PG^2 + 3FG^2+PG^2+DG^2$; and hence $4PG^2 + 3FG^2 + DG^2$ must be equal to a given space. Let FO be made the fourth part of DF, and join PO: then, by the *Lemma*, $3FG^2 + DG^2 = FD.DO + 4OG^2$. Wherefore $FD.DO + 4OG^2+4PG^2$, or $FD.DO+4PO^2$, is equal to a given space, and hence $4PO^2$, and PO itself, are given. Now the point O being given, P must lie in the circumference of a given circle.



COMPOSITION.

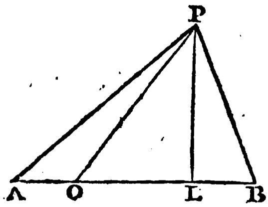
Bisect AB in E, trisect EC in F, and quadrisect FD in O; from the given space take away the accumulate rectangles $AB.BE+EC.CF+FD.DO$, and find (III. 33. El.) the side of a square equal to this difference: That straight line is the diameter of a circle, which is the *locus* required.

For join PE, PF, PO, and let fall the perpendicular PG upon DF; then $FD.DO+4PO^2=FD.DO+4OG^2+4PG^2= 3FG^2+DG^2+4PG^2=3FG^2+3PG^2+DP^2=3PF^2+DP^2$. Wherefore $AB.BE + EC.CF + 3PF^2 + DP^2$, is equal to the given space. But, from the composition of the last case, it is manifest that $AP^2 + BP^2 + CP^2 = AB.BE + EC.CF+3PF^2$; consequently $AP^2+BP^2+CP^2+DP^2$ are together equal to the given space.

By pursuing this mode of investigation, it is obvious that the proposition will be successively extended to any number of given points.

Scholium. The property now demonstrated is capable of being generalized. Thus, if any multiples of the squares of the

inflected lines, be together equal to a given space, the *locus* of their point of concourse is still a given circle: For, conceive so many points to be clustered together at each centre A, B, C, &c. of inflection, and the squares of the lines which proceed from them will evidently receive in effect a corresponding multiplication.—But the property may be traced out more clearly, and through all its shadings, by help of a simple extension of the *Lemma*. Let AP and BP be two straight lines inflected from the points A and B, and let the segment $OB = v.OA$; then, joining PO and drawing the perpendicular PL, it was proved that $v.AL^2 + BL^2 = AB.BO + (v+1)\ OL^2$; add $(v+1)\ PL^2$ to each, and $v(AL^2+PL^2)+BL^2+PL^2 = AB.BO+(v+1)(OL^2+PL^2)$, or $v.AP^2+BP^2=AB.BO+(v+1)OP^2$. Multiply both by n, and suppose $nv=m$, and there results $m.AP^2+n.BP^2 = n.AB.BO+(m+n)\ OP^2$. By repeated application of this principle, it may be demonstrated that $m.AP^2+n.BP^2+p.CP^2+q.DP^2$, &c. $=(m+n+p+q$, &c.$)\ OP^2$, together with certain multiples of given rectangles, and consequently that their point of concourse has for its *locus* a circle, whose centre is O and radius OP. But the property must likewise hold, if all those multiple squares were divided by the same number, that is, if instead of the squares of the inflected lines, there were substituted only similar rectilineal figures constructed upon them. If the given space should be equal to the rectangles, the circle will evidently contract to a point, and beyond this limit the problem becomes impossible. It is likewise obvious, that the centre O and radius OP will turn out the same, in whatever order the successive connected sections take place *.



* See Note LIX.

DEFINITION.

A *Porism* proposes to demonstrate that one or more things may be found, between which and innumerable other objects assumed after some given law, a certain specified relation is to be shown to exist.

The nature of a porism consists in affirming the possibility of finding such conditions, as will render a problem indeterminate, or capable of innumerable solutions.

PROP. XVIII. PORISM.

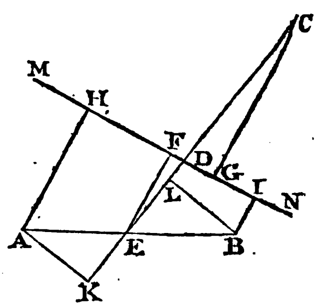
Three points being given, a fourth may be found, such that any straight line drawn through it shall have its distances from two of those equal to its distance from the third.

Let A, B, and C be given points, another point D may be found, so that, HDI being drawn through it, the perpendiculars AH and BI, let fall on the one side, shall be equal to CG on the other.

ANALYSIS.

Through the point D, draw CDK, and upon this let fall the perpendiculars AK, BL, and join AB, meeting KC in E.

Since CDK passes through C, its distances KA and LB on either side, from the two remaining points, must evidently be equal. Hence (I. 21. El.) the right-angled triangles AEK and BEL are equal, and consequently the side AE is equal to BE; wherefore E, being thus the point of bisection, is given. Draw the perpendicular EF; and it is evident (II. 10. El.) that 2EF = AH and BI. Now CG and EF being parallel, CD : DE :: CG : EF, and (V. 13. El.) CD : 2DE ::



CG : 2FE, or AH + BI; but, by hypothesis, CG = AH + BI, and therefore (V. 4. El.) CD = 2DE. Whence, CE being given, the point D is given.

COMPOSITION.

Bisect AB in E, join CE and trisect it in D; this is the point required.

For let fall the perpendicular EF. Because CG and EF are parallel, CD : DE : : CG : EF; but CD = 2DE, and therefore (V. 4. El.) CG = 2EF, that is, AH + BI.

The porism now demonstrated may be viewed as originating in the solution of this problem:—To draw, through the point M, a straight line MN, such that the perpendiculars AH and BI, let fall upon it from the points A and B, shall be together equal to the perpendicular CG, from the point C on the other side. The point D is found as before, and thence the position of MDN is assigned. But this straight line, it is evident, will become indeterminate if the point M should happen to coincide with D; on that supposition, the problem would admit of innumerable answers, or the diameter MDN might lie in every possible direction *.

PROP. XIX. PORISM.

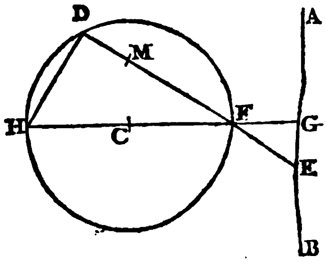
A circle and a straight line being given in position, a point may be found, such that any straight line, drawn through it and limited by these, shall contain a given rectangle.

Let the straight line AB, and the circle HDF, be given in position; it is required to determine a point F, which may divide any connecting straight line DFE into segments containing a rectangle that will be given.

* *See Note LX.*

ANALYSIS.

Through F draw HFG perpendicular to AB. By hypothesis, the rectangle HF.FG is likewise equal to the given space, and therefore equal to DF.FE; whence (V. 6. El.) DF : HF : : FG : FE, and the triangles DFH and GFE, having the vertical angles at F equal, are consequently similar, and the angle FDH is thus equal to FGE, or is a right angle. Wherefore HDF is a semicircle, of which HF is the diameter; but the centre C being given, the perpendicular HCG is thence given, and consequently the extremity of the diameter, or the point F. Again, the points H, F, and G being given, the rectangle under the segments HF and FG is given.



COMPOSITION.

From the centre C, let fall upon AB the perpendicular HCFG, cutting the circumference in F; this point has the property, that any intersecting line drawn through it will contain a given rectangle. For join DH, and the triangles FGE and FDH are similar; whence FG : FE : : FD : FH, and consequently FE.FD = FG.FH, which is manifestly given.

This porism also may be considered as arising out of the solution of a simple problem:—Through the point M, to draw a straight line DMFE, so that its segments DF and FE shall contain a given rectangle. The point F being found as before, DME is consequently given in position. But when the point M coalesces with F, the straight line DE can thus have no determinate position, or it will fulfil the conditions of the problem in whatever direction it be drawn.

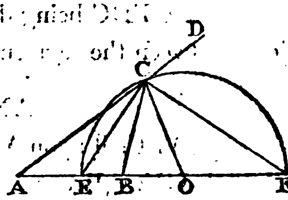
PROP. XX. PORISM.

A circle and a point being given, another point may be found, such that straight lines drawn from them to any point in the circumference, shall have a ratio which will be given.

The point B may be found, so that AC and BC, inflected to the given circumference ECF, shall have a ratio which may be likewise assigned.

ANALYSIS.

Draw AB, cutting the circle in E and F; join CE, CF, and produce AC. Because E, F are points in the circumference, AC : BC : : AE : EB, and AC : BC : : AF : FB; whence (VI. 11. cor. El.) CE bisects the vertical angle ACB, and CF the adjacent angle BCD; consequently the angle ECF, being the half of both of these, is a right angle, and (III. 22. El.) ECF, a semicircle.



Wherefore AF, thus passing through the centre O, is given in position. Now, since AF : FB : : AE : EB, alternately AF : AE : : FB : EB; hence EF, being cut externally and internally in the same ratio, EO is (VI. 7. El.) a mean proportional between AO and BO, or $EO^2 = AO.BO$. But AO and EO are given, and therefore BO and the point B are given. Again, because AO : EO : : EO : BO, by division and alternation, AE : EB : : EO : BO; that is, the inflected lines have the given ratio of EO to BO.

COMPOSITION.

Draw AF through the centre of the given circle, and make AO : EO : : EO : BO; B is the point required. For

join CO. Because EO is equal to CO, therefore AO : CO :: CO : BO; consequently the triangles ACO and CBO, having besides the common angle at O, are similar, and AC : AO :: BC : CO, or alternately AC : BC :: AO : CO, that is, in a given ratio.

The porism now demonstrated is evidently derived from the local theorem, which forms the 12th Proposition of this Book.

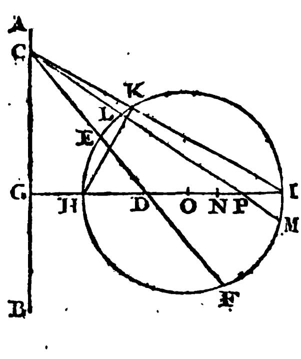
PROP. XXI. PORISM.

A circle and a straight line being given in position, a point may be found, such that any straight line drawn from it to the given line, shall be a mean proportional between the segments intercepted by the given circumference.

Let the straight line AB, and the circle HKF be given in position; it is possible to assign a point D, through which a straight line FDC being drawn, CD shall be a mean proportional between the segments CE and CF.

ANALYSIS.

From D let fall upon AB the perpendicular IDG, and join CI and HK. Because CE : CD :: CD : CF, $CD^2 = CE.CF =$ (III. 32. El.) CK.CI; and, since GI passes through the point D, GH : GD :: GD : GI, and $GD^2 = GH.GI$. But (II. 11. El.) $CD^2 = CG^2 + GD^2$, and consequently $CK.CI = CG^2 + GI.GH$; take these away from $CI^2 = CG^2 + GI^2$, and there remains $CI.KI = GI.HI$. Whence CI : GI :: HI : KI, and consequently the triangles CIG and HIK, having a com-



mon vertical angle, are similar. Wherefore the angle HKI, being thus equal to CGI, stands in a semicircle, of which HI is the diameter; consequently GI is given in position, and the points G, H, and I being thence given, the rectangle under GH and GI, or the square of GD, is given, and therefore the point D.

COMPOSITION.

Through the centre O, draw the perpendicular GOI; and find (VI. 18. El.) GD a mean proportional to GH and GI; D is the point required. For (III. 32. and II. 19. El.) $CE.CF = CO^2 - HO^2 = CG^2 + GO^2 - HO^2 = CG^2 + GH.GI$; but (V. 6. El.) $GD^2 = GH.GI$, and consequently $CE.CF = CG^2 + GD^2 = CD^2$.

This porism may be supposed to derive its origin from the problem:—"Through a given point P, in the diameter of a circle, to draw a straight line CLPM to the perpendicular AB, so that the rectangle under the segments CL and CM shall be equal to the square of GN." Since (III. 32. El.) $CL.CM = CK.CI = CI^2 - CI.KI$; but (II. 11. El.) $CI^2 = CG^2 + GI^2$, and $CI.KI = GI.HI$; whence $CL.CM = CG^2 + GI.GH$, or making $GD^2 = GI.GH$, $CL.CM = CG^2 + GD^2$ or CD^2, and consequently $CD^2 = GN^2$, or $CD = GN$. Wherefore the point D being given, the point C is also given, and thence the straight line CLPM. The problem then is solved by finding GD a mean proportional to GH and GI, and describing, from D with the radius GN, a circle to intersect the perpendicular in C. It is hence evident, that C is independent of the point P. Let CLM, therefore, coincide with CEF, and $CE.CF = GN^2 = CD^2$. But this property must evidently obtain, whatever be the position of the point C *.

PROP. XXII. PORISM.

A point being given in the diameter of a given circle, another point in the same extension may be

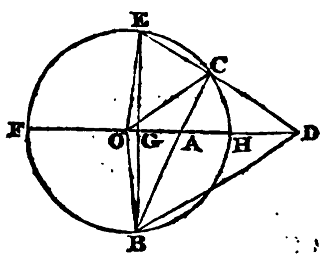
* See Note LXI.

found, such that the angle contained by two straight lines drawn from it to the extremities of a chord passing through the given point, shall be bisected by the diameter.

In the diameter FH of a given circle, let A be a given point through which any chord BAC is drawn; a point D may be found in the extension of the diameter, so that DC and DB being joined, the angle ADC shall be equal to *ADB*.

ANALYSIS.

Join EB, and draw EO and BO to the centre O. The triangles EOD and BOD, having the side EO equal to BO, OD common, and the angle ODE equal to ODB, and being likewise of the same affection, since the angles DEO and DBO are evidently both acute —are (I. 22. El.) equal, and consequently the angle EOG is equal to BOG. Whence



the triangles OEG and OBG are (I. 3. El.) also equal, and therefore EB is perpendicular to the diameter FH. Wherefore (VI. 9. El.) FA : AH : : FD : DH; but the ratio of FA to AH being given, and consequently that of FD to DH, the point D (VI. 6. El.) is given.

COMPOSITION.

Make (VI. 3. El.) OA : OH : : OH : OD, and then D is the point required. For join OC and OB. Because OH= OC, OA : OC : : OC : OD; wherefore the triangles AOC and COD, having thus the sides about their common angle DOC proportional, are similar; and hence the angle OCA is equal to ODC. In the same manner, it is proved that the angle OBA is equal to ODB. But BOC being an isosceles triangle, the angle OCA is equal to OBA; whence the angle ODC is equal to ODB.

This porism is likewise derived from the local theorem given in Prop. 12. For AC, DC, and AB, DB being inflected in the same ratio, AC : AB : : DC : DB; and consequently (VI. 11. cor. El.) the angle BDC is bisected by DA.

PROP. XXIII. PORISM.

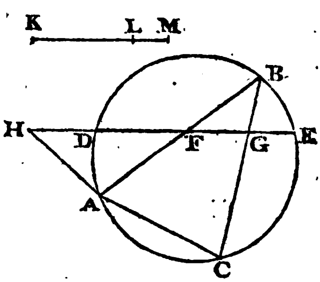
A point being given in the circumference of a circle, another point may be found, so that two straight lines inflected from them to the opposite circumference, shall cut off, on a given chord, extreme segments, whose alternate rectangles shall have a given ratio.

Let the circle ADBE, the point A, and the chord DE, be given in position,—another point C may be found, such that straight lines AB and CB inflected to the opposite circumference, shall form segments containing rectangles DG, FE, and DF, GE, in the ratio of KM to LM.

ANALYSIS.

Join CA, and produce it to meet the extension of the chord ED in H.

Because KM : LM : : DG.FE : DF.GE, by division KL : LM : : DG. FE—DF.GE : DF.GE; but DG.FE—DF.GE =(DF + FG) (GE + FG)—DF.GE=FG.DE, and consequently KL : LM : : FG.DE : DF.GE. Make KL : LM : : DE : DH, then KL : LM : : FG.DE : FG.DH; whence FG.DH=DF.GE, and, adding DF.FG to both, FH.FG= DF.FE=(III. 32. El.) AF.FB. Wherefore FH : FB : : AF : FG, and (VI. 14. El.) the triangles AFH and GFB are si-



milar, and consequently the angle AHF is equal to FBG; but the angle AHF is given, since the points A, H, and D are given, and, therefore, the chord AC, cutting off from the given circumference, a segment that contains a given angle ABC or FBG is given, and thence the point C.

COMPOSITION.

Produce the chord ED to H in the ratio of KM to LM, join HA, and, at any point B in the circumference, make the angle ABC equal to AHF; C is the point required.

For, the triangles AFH and GFB being evidently similar, FH : FB : : AF : FG, and FH.FG=FB.AF=DF.FE; whence FH.FG—DF.FG=DF.FE—DF.FG, or FG.DH= DF.GE. But KL : LM : : DE : DH : : FG.DE : FG.DH, and therefore KL : LM : : FG.DE : DF.GE; consequently (V. 9. El.) KM : LM : : FG.DE+DF.GE, or DG.FE : DF.GE.

The porism now investigated arises naturally out of this problem :—" From two given points A and C, one of which lies in a given circumference, to inflect straight lines AB and CB, so as to intercept on the chord DE segments that contain rectangles DG, FE and DF, GE, which are in a given ratio." For, the point H being assumed as before, the analysis requires that the angle ABC should be made equal to AHF. Whence, if on AC, a segment of a circle were described containing that angle, its contact or intersection with the given circumference, would determine the point of inflection. Supposing, therefore, the two circles entirely to coincide, the problem will in that case become indeterminate, or admit of innumerable answers.

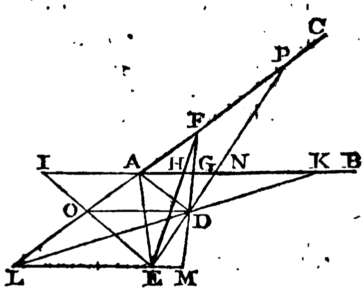
PROP. XXIV. PORISM.

Two points and two diverging lines being given in position, straight lines, inflected from those points to one of the diverging lines, intercept segments, on the other, from points that may be found, and containing a rectangle which will be likewise assignable.

Let DF and EF be inflected, from the points D and E, to the diverging line AC; they will cut off segments, on AB, from points I and K which may be found, so that the rectangle IH, GK shall be given.

ANALYSIS.

Join EI and EA, DA and DK, and produce ED to meet AC in P. Since A, F, and P are so many points of inflection, it is evident, from the hypothesis, that IA.AK=IH.GK=IN.NK; whence IH : IA : : AK : GK, and, by division, AH : IA : : AG : GK, and alternately AH : AG : : IA : GK. Through E, draw LEM parallel to AB and meeting AC and FD produced; then (VI. 2. El.) LE : LM : : AH : AG : : IA : GK.



Again, because IA.AK=IN.NK, IN : IA : : AK : NK, by division AN : IA : : AN : NK, and consequently IA=NK. Wherefore, by substitution, LE : LM : : NK : GK, and LE : EM : : NK : GN, or alternately LE : NK : : EM : GN, that is, (VI. 2. El.) ED : DN; hence (VI. 14. El.) the triangles LDE and KDN are similar, and LDK forms one single straight line. Join DO. Since IA=

NK, LE : IA : : LE : NK, that is, (VI. 2. El.) EO : OI : : ED : DN, and therefore (VI. 1. cor. I. El.) DO is parallel to AB. But the parallels OD and LM being given in position, the points O and L, and thence I and K, are given, and consequently the rectangle IA, AK is given.

COMPOSITION.

Draw DO, EL parallel to AB and meeting the extension of AC, join EO, LD, and produce them to meet AB in *I* and K; these are the points required. For DF and EF being inflected, LE : IA : : OE : OI : : ED : DN : : DM : DG : : LM : GK, and alternately LE : LM : : IA : GK; but LE : LM : : AH : AG, and therefore IA : GK : : AH : AG; consequently (V. 8. and 11. El.) IA : IH : : GK : AK, and IA.AK=IH.GK.

The porism thus investigated follows from this problem: "Two straight lines AB and AC being given in position, with the points I and K, E and D, to find a point F, such that the inflected lines EF and DF shall intercept segments IH and GK, containing a given space:" For, when the points I and K have the position before assigned, the construction becomes indeterminate.

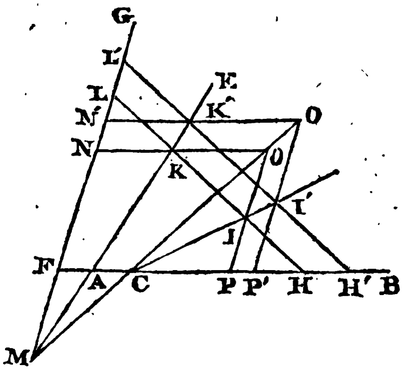
PROP. XXV. PORISM.

Three diverging lines being given in position, a fourth may be found, such that straight lines can be drawn intersecting all these and divided by them into proportional segments.

Let AB, CD, and AE be given diverging lines, and HIKL any transverse line cut by them in given ratios; a fourth diverging line FG may be found limiting the segment KL.

ANALYSIS.

Produce EA and GF to meet in M, through K and P draw NO and PO parallel to AB and FG, and meeting in O, join CO; let H′I′K′L′ be another transverse line divided into proportional segments, draw P′I′O′ parallel to PIO and meeting CO in O′, and join O′K′ and produce it to N′.



Because KO is parallel to PH, HI : IK : : PI : IO; and, since the parallels PO and P′O are cut by the diverging lines CP, CI, and CO, PI : IO : : P′I′ : I′O′; consequently H′I′ : I′K′ : : P′I′ : I′O′, and O′N′ is parallel to ON. Again, IK : KL : : OK : KN and I′K′ : K′L′ : : O′K′ : K′N′; wherefore OK : KN : : O′K′ : K′N′; and hence the straight lines OC, EA, and GF all converge to the same point M. Now CA : AF : : OK : KN : : IK : KL; whence the ratio of CA to AF being given, AF and the point F are given; but the point L is given, and, therefore, FLG is given in position.

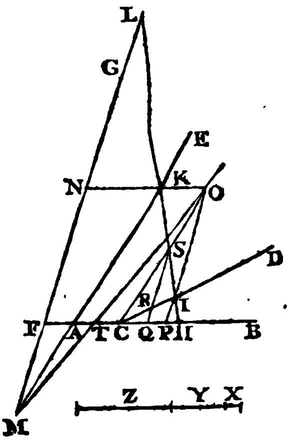
COMPOSITION.

Make CA to AF in the given ratio of the segment IK to KL, and join FL; this is the diverging line required. For draw NK and PI parallel to AB and FG, and meeting in O, join CO, and, assuming in it another point O′, draw likewise the parallels O′K′N′ and O′I′P′, intersecting AE and AB in K′ and I′; the transverse line H′I′K′L′ is cut similarly to HIKL.

For, since NO, N′O′ are parallel to AB, and OP, O′P′ parallel to FG, it follows that HI : IK : : PI : IO : : P′I′ : I′O′ : :

HT′ : T′K′. Again, because CA : AF : : IK : KL : : OK : KN; whence OC, EA, and GF converge to the same point, and consequently IK : KL : : OK : KN : : O′K′ : K′N′ : : I′K′ : K′L′.

The porism now demonstrated arises out of the indeterminate case of a celebrated problem:—"Four straight lines, AB, CD, AE and FG, being given in position, to draw a transverse line, HIKL, that shall be cut by them into segments in a given proportion." Suppose it done; produce GF and EA to meet in M, draw the parallels NKO and PIO, and join MTO. Because TA : AF : : OK : KN : : IK : KL, the ratio of TA to AF is given, and hence the point T and the straight line MO are given in position. Again, PI : IO : : HI : IK, and therefore the ratio of PI to PO is given; but the triangle CPI, being evidently given in species, the ratio of CP to PI is given; whence the ratio of CP to PR is given, and the triangle CPO is given in species. The straight lines MO and CO being, therefore, both given in position, their intersection O is given; consequently the parallels NO and PO are given in position, and thence are likewise given their intersections K and I, and the transverse line HIKL.



The construction is easily derived: For, having produced EA and GF to meet in M, make FA : AT : : Z : Y, and draw MTO. Again, take any point Q in CB, draw QS parallel to FG, and make QR : RS : : X : Y, join CS and produce it to meet MO in O, and draw OI and KO parallel to FG and AB; HIKL, which passes through the points of intersection I and K, is the straight line required. For HI : IK

:: PI : IO :: QR : RS :: X : Y, and IK : KL :: OK : KN :: TA : AF :: Y : Z.

Now, if the ratio of CA to AF should be the same as that of Y to Z, the point T will coincide with C, and the straight line TO with CO. The problem, therefore, becomes, in this case, porismatic, or every point whatever in CO has the property which belonged before to the single point O *.

DEFINITION.

Isoperimetrical figures are such as have equal perimeters, or the same extent of linear boundary.

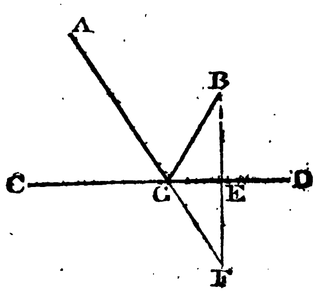
PROP. XXVI. PROB.

In a straight line given in position, to find a point, whose distances from two given points on the same side shall together be the least possible.

Let it be required, from the points A and B to some point in CD, to draw AG and BG, forming jointly a *minimum*.

ANALYSIS.

From B, either of the given points, let fall BE a perpendicular upon CD, and, having produced it equally on the opposite side, join GF. It is obvious that the triangles BEG, FEG are equal, and consequently that BG=GF; whence AG+GF is a *minimum*. But the points A and F are evidently both given, and since (I. 15. El.) the shortest communication between them is a straight line, its intersection G

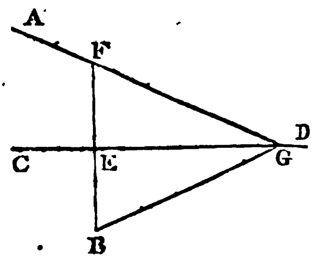


* See Note LXII.

with CD is given, and therefore the inflected lines AG and BG are given in position.

It hence appears that, when the combined distance of the points A and B from the straight line CD is the least possible, the incident angles AGC and BGD are equal.

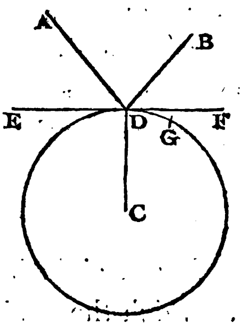
Cor. Hence also the solution of a similar problem :—" In a straight line given in position, to find a point the difference of whose distances from two given points shall be the greatest possible." If these points lie on the same side of the straight line CD, it is evident that the difference between AG and BG being (I. 16. El.) less than the base AB, this must be the extreme limit, or the difference must reach its *maximum* when AG and BG coincide with AB, and consequently the point G occurs where the production of AB meets CD.—But if A and B lie on opposite sides of CD, let fall the perpendicular BE which produce till EF be equal to it, and join AF and GF. The triangles BEG, FEG are evidently equal, and therefore BG=GF; but, in the triangle AFG, the difference of AG and GF, being less than AF, must attain its greatest extent, when that triangle is supposed to flatten into a straight line; in which case the angle AGE is equal to BGC.



PROP. XXVII. THEOR.

Straight lines drawn from two given points to the circumference of a given circle are the least possible, when they make equal angles with a tangent applied at the point of inflection.

Of all the straight lines inflected from the points A and B to the circumference of the circle GDH, AD and BD which meet the tangent EF at equal angles, form together a *minimum*.



For, by the last proposition, AD and BD, falling at an equal incidence, are jointly shorter than any other lines inflected from the points A and B to the straight line EF; but (I. 17. El.) such lines drawn to that tangent are less than the exterior lines which terminate in the circumference; whence, for both these reasons combined, AD and BD must form the *minimum* of all the straight lines inflected to the circumference GDH.

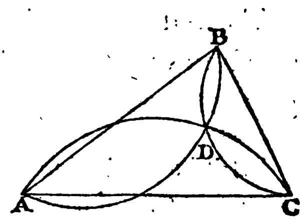
PROP. XXVIII. PROB.

To find a point, whose distances from three given points are the least possible.

Let it be required, from the points A, B, and C, to draw AD, BD, and CD, such that their sum shall be a *minimum*.

ANALYSIS.

If the distance BD were supposed to remain constant, the position of D, in the circumference of a circle described from B with the radius BD, must, by the last proposition, be such, when AD and CD compose a *minimum*, that the angle ADB shall be equal to CDB. For the same reason, if AD continued invariable, BD and CD, completing the *minimum*, must form with it equal angles ADB and ADC. Whence, uniting these conditions, the straight lines AD, BD, and CD all make equal angles about their point of concourse.



Hence this construction:—

Connect the triangle ABC, and upon each of the sides AC and BC describe equilateral triangles, and again circumscribe these by circles, which will intersect in the point D. For, the

angles ADC and CDB, being the supplements of angles of equilateral triangles, are each equal to two third parts of two right angles, or to one-third of four right angles; consequently three such angles will stand about the point D.

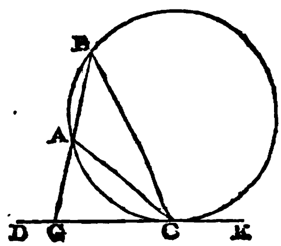
PROP. XXIX. PROB.

In a straight line given in position, to find a point, at which the straight lines, drawn to two given points on the same side, will contain the greatest angle.

Let it be required to draw AC and BC, so that the angle ACB shall be a *maximum*.

ANALYSIS.

Describe a circle about the points C, A, and B. Because the angle ACB is greater than any other which has its vertex in DE, the circumference must lie within that straight line, and therefore DE touches the circle.



It is hence evident, that $GA.GB = GC^2$, and, therefore, the point C is assigned.

PROP. XXX. PROB.

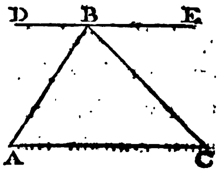
To find a triangle with a given perimeter, and standing on a given base, which shall contain the greatest area.

Let it be required to find a triangle ABC, constituted on the base AC, and containing, within a given perimeter, the greatest possible surface.

ANALYSIS.

Since the base of the triangle ABC is constant while its area forms a *maximum*, the corresponding altitude must evi-

dently be the greatest possible, and consequently the vertex B must lie in a parallel the remotest from AC. Supposing, therefore, the parallel DE to retain its place, the sum of the sides AB and CB, and consequently the whole perimeter of the triangle, will, by Proposition 26. of this Book, be the least possible, when the angle ABD is equal to CBE. Whence, preserving the same perimeter, the parallel will be enabled to recede to the greatest distance from AC, if these incident angles still maintain their equality; but DE being parallel to AC, the alternating angles BAC and BCA (I. 23. El.) are likewise equal, and consequently their opposite sides CB and AB. The triangle ABC is thus isosceles; and it is also given, for its sides are all given.



Cor. Hence an equilateral polygon is that which, under a given number of sides, contains, within the same perimeter, the greatest possible surface: For, the rest of the figure remaining constant, suppose any two adjacent sides to vary, and the accrescent triangle so formed will, by this proposition, be a *maximum*, when those sides are equal. The polygon, deriving its expansion from the aggregate of the exterior triangles, must therefore be the greatest possible, when such triangles are in every combination isosceles, and consequently all the sides of the figure equal.

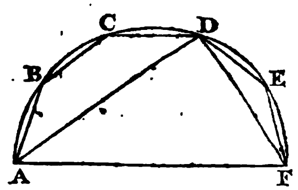
PROP. XXXI. THEOR.

If a polygon have all its sides given, except one,—it will contain the greatest area, when it can be inscribed in a semicircle, of which that indeterminate side is the diameter.

Let the polygon ABCDEF, having given sides AB, BC, CD, DE and EF, stand upon a base AF, which is variable;

the area will attain its *maximum*, when AF becomes the diameter of a circumscribing semicircle.

For, AD and FD being inflected to any point D, the spaces ABCD and DEF will evidently remain the same, while the angle ADF is enlarged, or the points A and F are distended. Whence the polygon must contain the greatest area, when the included triangle ADF contained by given sides AD and DF, is a *maximum*. Now, this will take place when *the altitude* of the triangle, or the perpendicular let fall from the vertex F upon AD, is the greatest possible. Wherefore (I. 18. El.) ADF is a right angle, and consequently (III. 22. El.) the point D lies in a semicircumference. But the same reason applies to every other intermediate point B, C, or E, of the polygon, which consequently, in its state of *maximum*, is disposed within a semicircle described on the variable side AF.



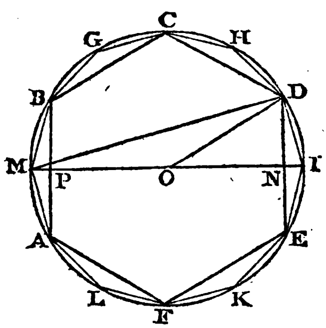
Cor. 1. Hence a polygon, whose sides are all *given*, contains the greatest area, when it can be inscribed *in* a circle. For let ABCD be a polygon, which has each of its sides AB, BC, CD, and AD given. Draw the diameter AF, and join DF. The polygon ABCDF is thus a *maximum*; but the triangle ADF being evidently determinate, the remaining polygon ABCD is likewise a *maximum*.

Cor. 2. Hence a regular polygon is that which, with a given perimeter, formed by a given number of sides, contains the greatest area. For, by the corollary to the last Proposition, the sides are all equal; but its angles are (III. 14. and 18. El.) also equal, since it occupies the circumference of a circle.

PROP. XXXII. THEOR.

A circle contains, within a given perimeter, the greatest possible area.

From the preceding investigations, it appears, that the perimeter and number of sides being given, the figure of greatest capacity is a regular polygon. Let ABCDEF be such a polygon, bounded by the given perimeter: Bisect the corresponding arcs of the circumscribing circle, and another regular polygon MBGCHDIEKFLA will arise, having twice the number of sides. Draw the diameter MI, and join MD and OD. Both polygons are alike composed of triangles equal to ODN and ODI, and consequently the area of the polygon ABCDEF is to that of MBGCHDIEKFLA as ON to OI, or as 2ON or PN to 2OI or MI. But if this exterior polygon MBGCHDIEKFLA were contracted to the same perimeter with ABCDEF, its area would (VI. 26. El.) be diminished in the ratio of DI^2 to DN^2, that is, (III. 22. and VI. 16. cor. 1. El.) in the ratio of the rectangle MI, NI to MN, NI, or that of MI to MN. Whence (V. 16. El.) the original polygon is to another of equal perimeter and with double the number of sides, as PN to MN. An isoperimetrical figure thus has its area always increased, by doubling the number of its sides. Continuing this duplication, therefore, the regular polygons which arise in succession will have their capacity perpetually enlarged. Whence the circle, as it forms the limit or extreme boundary of all those polygons, must, with a given circumference, contain the greatest possible space *.



* See Note LXIII.

ELEMENTS

OF

PLANE TRIGONOMETRY.

ELEMENTS

OF

PLANE TRIGONOMETRY.

TRIGONOMETRY is the science of calculating the sides or angles of a triangle. It grounds its conclusions on the application of the principles of Geometry and Arithmetic.

The sides of a triangle are measured, by referring them to some definite portion of linear extent, which is fixed by convention. The mensuration of angles is effected, by means of that universal standard derived from the partition of a circuit. Since angles were shown to be proportional to the intercepted arcs of a circle described from their vertex, the subdivision of the circumference therefore determines their magnitude. A quadrant, or the fourth part of the circumference, as it corresponds to a right angle, hence forms the basis of angular measures. But these measures depend on the relation of certain orders of lines connected with the circle, and which it is necessary previously to investigate.

DEFINITIONS.

1. The *complement* of an arc is its defect from a quadrant; and its *supplement* is its defect from a semicircumference.

2. The *sine* of an arc is a perpendicular let fall from one of its extremities upon a diameter passing through the other.

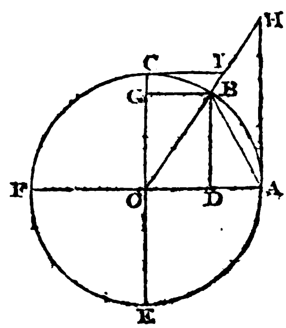
3. The *versed sine* of an arc is that portion of a diameter intercepted between its sine and the circumference.

4. The *tangent* of an arc is a perpendicular drawn at one extremity to a diameter, and limited by a diameter extending through the other.

5. The *secant* of an arc is a straight line which joins the centre with the termination of the tangent.

In naming the *sine*, *tangent*, or *secant* of the *complement* of an arc, it is usual to employ the abbreviated terms of *cosine*, *cotangent*, and *cosecant*. A farther contraction is frequently made in noting the radius and other lines connected with the circle, by retaining only the first syllable of the word, or even the mere initial letter.

Let ACFE be a circle, of which the diameters AF and CE are at right angles; having taken any arc AB, produce the radius OB, and draw BD, AH perpendicular to AF, and BG, CI perpendicular to CE. Of this assumed arc AB, the *complement* is BC, and the *supplement* BCF; the *sine* is BD, the *cosine* BG or OD, the *versed sine* AD, the *coversed sine* CG, and the *supplementary versed sine* FD; the *tangent* of AB is AH, and



its *cotangent* CI; and the *secant* of the same arc is OH, and its *cosecant* OI.

Several obvious consequences flow from these definitions:—

1. Since the diameter which bisects an arc bisects also the chord at right angles, it follows that half the chord of any arc is equal to the sine of half that arc.

2. In the right-angled-triangle ODB, $BD^2+OD^2=OB^2$; and hence the squares of the sine and cosine of an arc are together equal to the square of the radius.

3. The triangle ODB being evidently similar to OAH, OD : DB :: OA : AH; that is, the cosine of an arc is to the sine, as the radius to the tangent.

4. From the similar triangles ODB and OAH, OD : OB :: OA : OH; wherefore the radius is a mean proportional between the cosine and the secant of an arc.

5. Since $BD^2=AD.FD$, it is evident that the sine of an arc is a mean proportional between the versed sine and the supplementary versed sine, or between the sum and difference of the radius and the cosine.

6. Hence also the chord of an arc is a mean proportional between the versed sine and the diameter; for $AB^2 = AD.AF$.

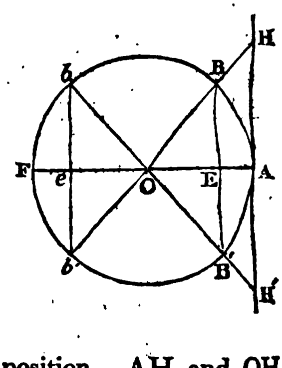
7. The triangles OAH and ICO being similar, AH : OA :: OC : CI; and hence the radius is a mean proportional between the tangent of an arc and its cotangent.

8. Since $OD^2=BG^2=CG.GE$, it follows that the cosine of an arc is a mean proportional between the sum and the difference of the radius and the sine.

The circumference of the circle is commonly divided into 360 equal parts, called degrees, each of them being subdivided into 60 minutes, and these again being each

distinguished into 60 seconds. It very seldom is required to carry this subdivision any farther. Degrees, minutes, seconds, or thirds, are conveniently noted by these marks, ° ′ ″ ‴ Thus, 23° 27′ 43″ 42‴, signifies 23 degrees, 27 minutes, 43 seconds, and 42 thirds *.

Scholium. To discern more clearly the connexion of the lines derived from the circle, it will be proper to trace their successive values, while the corresponding arc is supposed to increase. Let the arc AB′, on the opposite side, be made equal to AB, draw the diameter FOA, extend the diameters b′OB and bOB′, join BB′ and bb′, and at A apply the double tangent HAH′. It is evident that BE=be, or that the sine of the arc AB is equal to the sine of its supplement ABb. But B′E and b′e, or the sines of ABFb′ and ABFb′B′, which lie on the opposite side of the diameter, are likewise equal to BE; that is, the inverted sine of an arc is equal to the sine of that arc or of its supplement, augmented, each by a semicircumference. The arc AB, and its defect ABFB′ from a whole circumference, have both the same cosine OE; and the supplemental arc ABb, and its defect from a whole circumference, have likewise the same cosine, although with an inverted position. AH and OH are respectively the tangent and secant not only of AB, but of the arc ABbFb′, which is compounded of the original arc and a semicircumference; and the similar lines AH′ and OH′, on the opposite side, are at once the tangent and secant of the supplementary arc ABb, and of ABbFb′B′, likewise compounded of that arc and a semicircumference.



As the prolonged diameter b′OBH, therefore, turns about

* See Note LXIV.

the centre, the sine and tangent both increase, till the arc attains 90°, when the sine becomes equal to the radius, and the tangent vanishes into unlimited extent. Between 90° and 180°, the sine again diminishes, and the tangent, re-appearing in the opposite direction, likewise contracts by successive diminutions. In the third quadrant, the sine emerges with a contrary position, and increases till it becomes equal to the radius; while the tangent, resuming its first position, stretches out till it vanishes away. Between 270° and 360°, the opposite sine again contracts, and the tangent, re-appearing on the same side, shrinks also by degrees to a point. In the first and fourth quadrants, the cosine lies on the same side of the centre, while the secant stretches from it in the direction of the extremity of the arc; but, in the second and third quadrants, the cosine shifts to the opposite side, and the secant shoots from the centre in a direction opposite to the termination of the arc.

The same phases are thus repeated at each succeeding revolution. Hence, if m denote any integral number, the sine of an arc a is equal to the sine of the arc $(2m-1)\,180°-a$, and to the opposite sines of $(2m-1)\,180°+a$ and of $2m.180°-a$; the cosine and secant of an arc a are equal to the cosine and secant of $2m.180°-a$, and to the opposite cosines and secants of $(2m-1)\,180°-a$ and of $(2m-1)\,180°+a$; and the tangent or cotangent of an arc a is equal to the tangent or cotangent of the arc $(2m-1)\,180°+a$, and to the opposite tangents or cotangents of the arcs $(2m-1)\,180°-a$ and $2m.180°-a$.

An arc may, by a simple extension of analogy, be conceived to comprehend innumerable other arcs. Thus, the arc AB, in fact, represents all the arcs which have their origin at A and their termination at B; it therefore includes not only the small arc AB, but that arc as augmented by successive revolutions, or the repeated addition of entire circumferences. Hence the sine or tangent of an arc a are the same with the sine or tangent of any arc $n.360°+a$ *.

* See Note LXV.

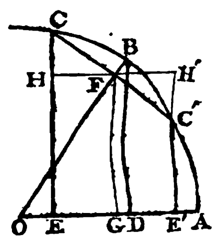
PROP. I. THEOR.

The rectangle under the radius and the sine of the sum or difference of two arcs, is equal to the sum or difference of the rectangles under their alternate sines and cosines.

Let A and B denote two arcs, of which A is the greater; then, $R \times sin(A \pm B) = sin A \times cos B \pm cos A \times sin B$.

For, having made BC′=BC, it is evident that AC and AC′ will represent the sum and difference of the arcs AB and BC; join OB and CC′, and draw HFH′ parallel, and CE, FG, BD, and H′CE perpendicular, to the radius OA.

The triangles COF and C′OF, having the side CO equal to C′O, OF common, and the contained angles FOC and FOC′ measured by the equal arcs BC and BC, are equal; wherefore OF bisects CC′ at right angles. But the triangles OBD and OFG being similar, OB : BD : : OF : FG, or HE, and consequently OB.HE=BD.OF. The triangles OBD and CFH are likewise similar, for the right angle CFO being



equal to HFG, if HFO be taken from both, the remaining angle CFH is equal to OFG or OBD; whence OB : OD : : CF : CH, and OB.CH=OD.CF. Wherefore OB.HE + OB.CH, or OB.CE=BD.OF+OD.CF. But BD and OD are the sine and cosine of the arc AB, CF and OF the sine and cosine of BC, and CE is the sine of the compound arc AC. Consequently, $R \times sin AC = sin AB \times cos BC + cos AB \times sin BC$.

Again, taking the difference of the rectangles OB, H′E′ and OB, C′H′, and OB × C′E′ = BD × OF—OD × CF; whence $R \times sin AC' = sin AB \times cos BC - cos AB \times sin BC$.

Cor. 1. If the two arcs A and B be equal, it is obvious that $R \times sin2A = sinA \times 2cosA$.

Cor. 2. Let the arc A contain 45°; then $R \times sin(45° \pm B) = sin45°(cosB \pm sinB) = \sqrt{\frac{1}{2}R^2}(cosB \pm sinB)$, or $sin(45° \pm B) = \sqrt{\frac{1}{2}}(cosB \pm sinB.)$

Cor. 3. Let $2A = C$, and, by the first corollary, $R \times sinC = sin\frac{1}{2}C \times 2cos\frac{1}{2}C$.

PROP. II. THEOR.

The rectangle under the radius and the cosine of the sum or difference of two arcs, is equal to the difference or the sum of the rectangles under their respective cosines and sines.

Let A and B denote two arcs, of which A is the greater; then $R \times cos(A \pm B) = cosA \times cosB \mp sinA \times sinB$.

For, in the preceding figure, the triangles OBD and OFG being similar, OB : OD : : OF : OG, and OB.OG = OD.OF, and the triangles OBD and CFH being likewise similar, OB : BD : : CF : FH, or GE, and consequently OB.GE = BD.CF. Wherefore OB.OG−OB.GE = OB.OE = OD.OF−BD.CF; that is, $R \times cosAC = cosAB \times cosBC - sinAB \times sinBC$.

Again, taking the sum of those rectangles, $OB \times OG + OB \times GE' = OB \times OE' = OD \times OF + BD \times CF$; whence $R \times cosAC' = cosAB \times cosBC + sinAB \times sinBC$.

Cor. 1. If A and B represent two equal arcs, it will follow, that $R \times cos2A = cosA^2 - sinA^2 = (cosA + sinA)(cosA - sinA)$; or, since $cosA^2 = R^2 - sinA^2$, $R \times cos2A = R^2 - 2sinA^2 = 2cosA^2 - R^2$.

Cor. 2. Hence $sinA^2 = \frac{1}{2}R(R - cos2A)$, and $cosA^2 = \frac{1}{2}R(R + cos2A)$; wherefore $sinA^2 - sinB^2 = \frac{1}{2}R(cos2B - cos2A)$.

Cor. 3. Let the arc be equal to 45°, and $R \times cos(45° \pm B) = sin\ 45°(cos B \mp sin B)$.

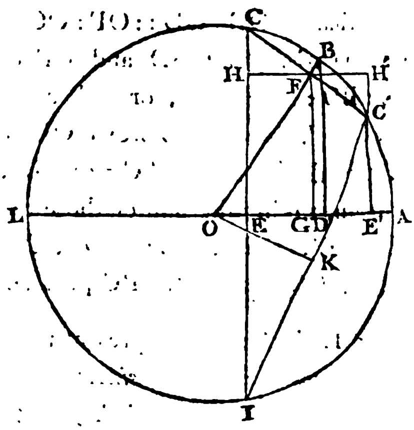
Cor. 4. Let $2A = C$, and by the first corollary, $R \times cos C = R^2 - 2 sin \tfrac{1}{2}C^2 = 2 cos \tfrac{1}{2}C^2 - R^2$.

PROP. III. THEOR.

Of three equidifferent arcs, the rectangle under the radius and the sum or difference of the *sines* of the extremes, is equal to twice the rectangle under the cosine or sine of the common difference and the sine or cosine of the mean arc.

Let A—B, A, and A+B represent three arcs increasing by the difference B; then $R\left(sin(A+B) + sin(A-B)\right) = 2 cos B \times sin A$, and $R\left(sin(A+B) - sin(A-B)\right) = 2 sin B \times cos A$.

These properties are easily deduced by combining the preceding theorems; but they are more easily perceived, by referring immediately to the original figure. The triangles OBD and OFG being similar, OB : BD :: OF : FG, or OB : BD :: 2OF : 2FG or CE+C′E′, and $OB(CE+C'E') = 2OF \times BD$; that is, $R(sin AC + sin AC') = 2 cos BC \times sin AB$.



Again, OB : OD : : CF : CH : : 2CF : 2CH or CE—C′E′, and $OB(CE - C'E') = 2CF \times OD$; consequently $R(sin AC - sin AC') = 2 sin\ BC \times cos AB$.

Cor. 1. Hence also $R\big(\cos(A-B)+\cos(A+B)\big)=2\cos B\times\cos A$, and $R\big(\cos(A-B)-\cos(A+B)\big)=2\sin B\times\sin A$.

For $OB : OD :: OF : OG :: 2OF : 2OG$ or $OE'+OE$, and $OB(OE'+OE)=2OF\times OD$; that is, $R(\cos AC'+\cos AC)$ $=2\cos BC\times\cos AB$. Again, $OB : BD :: CF : FH :: 2CF : 2FH$, or $OE'-OE$, and $OB(OE'-OE)=2CF\times BD$; that is, $R(\cos AC'-\cos AC)=2\sin BC\times\sin AB$.

Cor. 2. Let the radius be expressed by unit, and arcs B and A, denoted by a and na; then collectively $2\sin a\times\cos na$ $=\sin(n+1)a-\sin(n-1)a$, $2\cos a\times\sin na=\sin(n+1)a+\sin(n-1)a$, $2\sin a\times\sin na=\cos(n-1)a-\cos(n+1)a$, and $2\cos a\times\cos na=\cos(n-1)a+\cos(n+1)a$.

Cor. 3. Since $\text{vers}B=R-\cos B$, it follows that $R\big(\sin(A+B)$ $+\sin(A-B)\big)=2R\times\sin A-2\text{vers}B\times\sin A$, and consequently $R\times\sin(A+B)=2R\times\sin A-R\times\sin(A-B)-2\text{vers}B\times\sin A$, or $R(\sin(A+B)-\sin A)=R\big(\sin A-\sin(A-B)\big)-2\text{vers}B\times\sin A$. In the same way, it may be shown that $R(\cos(A-B)-\cos A)=$ $R\big(\cos A-\cos(A+B)\big)-2\text{vers}B\times\cos A$.

Cor. 4. If the mean arc contain 60°; then $R\big(\sin(60°+B)$ $-\sin(60°-B)\big)=2\sin B\times\cos 60°$, or $\sin B\times 2\sin 30°$. But twice the sine of 30° being (cor. 1. def.) equal to the chord of 60° or the radius, it is evident that $\sin(60°+B)-$ $\sin(60°-B)=\sin B$, or $\sin(60°+B)=\sin(60°-B)+\sin B$.— This property also follows from Prop. 14. Book IV. of the Elements; for $BD=AD+CD$, and $\frac{1}{2}BD=\frac{1}{2}AD+\frac{1}{2}CD$, or $\sin\frac{1}{2}BAD=\sin\frac{1}{2}AD+\sin\frac{1}{2}CD$, that is, $\sin(60°+\frac{1}{2}AD)=$ $\sin\frac{1}{2}AD+\sin(60°-\frac{1}{2}AD.)$

Cor. 5. Produce CE to the circumference, join C′I meeting the production of FG in K, and join OK. Since FK is parallel to CI and bisects CC′, it likewise bisects IC′; and hence OK is perpendicular to KC′, which is, therefore, the sine of half the arc IAC′, or of half the sum of the arcs AC and AC′, as CF is the sine of half their difference. But

(II.24.El.) $IC'^2 - CC'^2 = IC \times 2C'F'$, or $C'K^2 - CF^2 = CE \times CE'$; consequently $\sin^2 AB - \sin^2 BC = \sin AC \times \sin AC'$, or, employing the general notation, $\sin A^2 - \sin B^2 = \sin(A+B) \times \sin(A-B) =$ (2. cor. 2.) $\frac{1}{2}R(\cos 2B - \cos 2A)$ *.

Scholium. By help of this proposition, the sines and cosines of multiple arcs are easily determined; but the expressions for them will become simpler, if, as in cor. 2. the radius be supposed equal to unit. For A, 2A and 3A being three equidifferent arcs, $\sin A + \sin 3A = 2\cos A \times \sin 2A = 2\cos A \times 2\cos A \times \sin A$, or $\sin 3A = 4\cos A^2 \times \sin A - \sin A$; and $\cos A + \cos 3A = 2\cos A \times \cos 2A = 2\cos A\,(2\cos A^2 - 1) = 4\cos A^3 - 2\cos A$, or $\cos 3A = 4\cos A^3 - 3\cos A$. Again, since 2A, 3A, and 4A are equidifferent arcs, $\sin 2A + \sin 4A = 2\cos A \times \sin 3A = 8\cos A^3 \times \sin A - 2\cos A \times \sin A$, or $\sin 4A = 8\cos A^3 \times \sin A - 4\cos A \times \sin A$; $\cos 2A + \cos 4A = 2\cos A \times \cos 3A = 2\cos A(4\cos A^3 - 3\cos A)$, or $\cos 4A = 8\cos A^4 - 8\cos A^2 + 1$. In like manner, assuming the equidifferent arcs 3A, 4A, 5A, and the sine and cosine of 5A are found; and this mode of procedure may be continually repeated. To abridge the notation, however, it will be proper to express the sine and the cosine of the arc a, by s and c. The results are thus expressed in a tabular form:

$$
(1.)\quad
\begin{aligned}
\mathit{Sin}\ 2a &= 2cs.\\
\mathit{Sin}\ 3a &= 4c^2s - s.\\
\mathit{Sin}\ 4a &= 8c^3s - 4cs.\\
\mathit{Sin}\ 5a &= 16c^4s - 12c^2s + s.\\
\mathit{Sin}\ 6a &= 32c^5s - 32c^3s + 6cs.\\
\mathit{Sin}\ 7a &= 64c^6s - 80c^4s + 24c^2s - s.
\end{aligned}
$$

&c. &c. &c.

$$
(2.)\quad
\begin{aligned}
\mathit{Cos}\ 2a &= 2c^2 - 1.\\
\mathit{Cos}\ 3a &= 4c^3 - 3c.\\
\mathit{Cos}\ 4a &= 8c^4 - 8c^2 + 1.\\
\mathit{Cos}\ 5a &= 16c^5 - 20c^3 + 5c.\\
\mathit{Cos}\ 6a &= 32c^6 - 48c^4 + 18c^2 - 1.
\end{aligned}
$$

&c. &c. &c.

* See Note LXVI.

If in these expressions, $1-s^2$ be substituted for c^2, in the sines of the odd multiples of a, and in the cosines of the even multiples;—the sines and cosines of such multiple arcs will be represented merely by the powers of the sine a.

$$
(3.)\quad
\begin{aligned}
Sin\ 3a &= 3s - 4s^3.\\
Sin\ 5a &= 5s - 20s^3 + 16s^5.\\
Sin\ 7a &= 7s - 56s^3 + 112s^5 - 64s^7.
\end{aligned}
$$

&c. &c. &c.

$$
(4.)\quad
\begin{aligned}
Cos\ 2a &= +1 - 2s^2.\\
Cos\ 4a &= +1 - 8s^2 + 8s^4.\\
Cos\ 6a &= +1 - 18s^2 + 48s^4 - 32s^6.
\end{aligned}
$$

&c. &c. &c.

If the terms of the first table be repeatedly multiplied by $2s$, and those of the second by $2c$, observing the substitutions of cor. 2, there will result expressions for the sines and cosines. Thus, $2sin\ a^2 = 2s.s = -cos\ 2a + 1$, $4sin\ a^3 = -2s.cos\ 2a + 2s = -sin\ 3a + sin\ a + 2s = -sin\ 3a + 3s$, and $8sin\ a^4 = -2s.sin\ 3a + 2s.3s = +cos\ 4a - cos\ 2a - 3cos\ 2a + 3 = cos\ 4a - 4cos\ 2a + 3$. Again, $2cos\ a^2 = 2c.c = cos\ 2a + 1$, $4cos\ a^3 = 2c.cos\ 2a + 2c = cos\ 3a + cos\ a + 2c = cos\ 3a + 3cos\ a$, and $8cos\ a^4 = 2c.cos\ 3a + 2c.3cos\ a = cos\ 4a + cos\ 2a + 3cos\ 2a + 3 = cos\ 4a + 4cos\ 2a + 3$. In this manner, the following tables are formed.

$$
(5.)\quad
\begin{aligned}
Sin\ a &= s.\\
2\ Sin\ a^2 &= -cos\ 2a + 1.\\
4\ Sin\ a^3 &= -sin\ 3a + 3s.\\
8\ Sin\ a^4 &= +cos\ 4a - 4cos\ 2a + 3.\\
16\ Sin\ a^5 &= +sin\ 5a - 5sin\ 3a + 10s.\\
32\ Sin\ a^6 &= -cos\ 6a + 6cos\ 4a - 15cos\ 2a + 10.\\
64\ Sin\ a^7 &= -sin\ 7a + 7sin\ 5a - 21sin\ 3a + 35s.
\end{aligned}
$$

&c. &c. &c.

$$\begin{aligned}
Cos\ a &= c.\\
2\ Cos\ a^2 &= cos\ 2a + 1.\\
4\ Cos\ a^3 &= cos\ 3a + 3c.\\
(6.)\quad 8\ Cos\ a^4 &= cos\ 4a + 4cos\ 2a + 3.\\
16\ Cos\ a^5 &= cos\ 5a + 5cos\ 3a + 10c.\\
32\ Cos\ a^6 &= cos\ 6a + 6cos\ 4a + 15cos\ 2a + 10.\\
64\ Cos\ a^7 &= cos\ 7a + 7cos\ 5a + 21cos\ 3a + 35c.
\end{aligned}$$

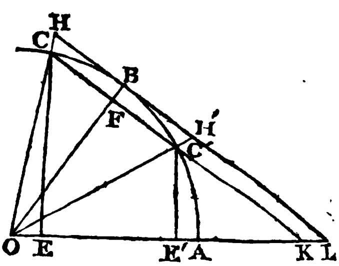
&c. &c. &c *.

PROP. IV. THEOR.

The sum of the sines of two arcs is to their difference, as the tangent of half the sum of the arcs to the tangent of half the difference.

If A and B denote two arcs; the $sin A + sin B : sin A - sin B :: tan \frac{A+B}{2} : tan \frac{A-B}{2}$.

For, let AC and AC′ be the sum and difference of the arcs AB and BC, or BC′; draw the perpendiculars CE, and C′E′, extend the chord CC′, and apply at B the parallel tangent HBL, meeting in K and L the diameter produced, and draw OCH, OFB and OC′H′. Because CE is parallel to C′E′, and CK to HL, CE : C′E′ :: CK : C′K (VI. 2. El.) HL : H′L; and consequently CE+C′E′ : CE−C′E′ :: HL+H′L : HL−HL′, that is, 2BL : 2BH, or BL : BH. But CE and C′E′ are the sines of the arcs AC and AC′, and BL and BH are the tangents of AB and BC, or of half the sum and half the difference of those arcs.



Wherefore $sin AC + sin AC' : sin AC - sin AC' :: tan \frac{AC+AC'}{2} : tan \frac{AC-AC'}{2}$.

* See Note LXVII.

Cor. 1. The sines of the sum and difference of two arcs are proportional to the sum and difference of their tangents. For CE : C'E' :: HL, or BL+BH : H'L, or BL—BH; that is, resuming the general notation, $sin(A+B) : sin(A-B) :: tanA+tanB : tanA-tanB$.

Cor. 2. Let the greater arc be equal to a quadrant; and $R+sinB : R-sinB :: tan(45°+\frac{1}{2}B) : tan(45°-\frac{1}{2}B)$, or $cot(45°+\frac{1}{2}B)$. But, the radius being a mean proportional between the tangent and cotangent of any arc; and the cosine of an arc being a mean proportional between the sum and difference of the radius and the sine, it follows that $R+sinB : cosB :: R : tan(45°-\frac{1}{2}B)$, and $R-sinB : cosB$, or $cosB : R+sinB, :: R : tan(45°+\frac{1}{2}B)$.

Or, if instead of B, there be substituted its complement, these analogies will become $R+cosB : sinB :: R : tan\frac{1}{2}B$, and $R-cosB : sinB :: R : cot\frac{1}{2}B$.

Cor. 3. Since $cosB : R :: R-sinB : tan(45°-\frac{1}{2}B)$, and $cosB : R :: R+sinB : tan(45°+\frac{1}{2}B)$, therefore (V. 19. El.) $cosB : R :: 2R : tan(45°-\frac{1}{2}B)+tan(45°+\frac{1}{2}B)$; that is, supposing B to be the complement of 2C, $sin2C : 2R :: R : tanC+cotC$. But (Prop. 1. cor. 1.) $R\times sin2C=2cosC\times sinC$, and consequently $cosC\times sinC : R^2 :: R : tanC+cotC$.

Cor. 4. Since (4. cor. def.) $cosB : R :: R : secB$, and (3. cor. def.) $cosB : sinB :: R : tanB$, therefore $cos B : R+sinB :: R : tanB+secB$, and consequently (2. cor. def.) $tan(45°+\frac{1}{2}B)= tanB+secB$.—This also appears clearly from the figure, on supposing OH'=H'L', or the angle LOH' equal to QLH', and consequently the arc AC' equal to the complement of AB.

PROP. V. THEOR.

As the difference or sum of the square of the radius and the rectangle under the tangents of two arcs, is to the square of the radius,—so is the sum or difference of their tangents, to the tangent of the sum or difference of the arcs.

Let A and B denote two arcs, of which A is the greater; then $R^2 \mp tanA \times tanB : R^2 :: tanA \pm tanB : tan(A \pm B)$.

For (3. cor. def.) $R : tanA :: cosA : sinA$, and $R : tanB :: cosB : sinB$; whence (V. 22. El.) $R^2 : tanA \times tanB :: cosA \times cosB : sinA \times sinB$, and (V. 11. and 7. El.) $R^2 \mp tanA \times tanB : R^2 :: cosA \times cosB \mp sinA \times sinB : cosA \times cosB$, that is, $R^2 \mp tanA \times tanB : R^2 :: R \times cos(A \pm B) : cosA \times cosB$. But (3. cor. def.) $cos(A \pm B) \times tan(A \pm B) = R \times sin(A \pm B)$, and $cos\,A \times cos\,B\ (tan\,A \pm tan\,B) = cos\,A \times tan\,A \times cos\,B \pm cosA \times cosB \times tanB = R \times sinA \times cosB \pm R \times cosA \times sinB =$ (Prop. 1.) $R^2 \times sin(A \pm B)$; wherefore (V. 6. and 3. El.) $R \times cos(A \pm B) : cosA \times cosB :: tanA \pm tanB : tan(A \pm B)$, and consequently $R^2 \mp tanA \times tanB : R^2 :: tanA \pm tanB : tan(A \pm B)$ *.

Cor. 1. Let the two arcs be equal; and $R^2 - tanA^2 : R^2 :: 2tanA : tan2A$.

Cor. 2. Let the greater arc contain 45°; and since $tan45° = R$, it follows that $R^2 \mp R \times tanB : R^2 :: R \pm tanB : tan(45° \pm B)$, or $R \mp tanB : R \pm tanB :: R : tan(45° \pm B)$ †.

Scholium. Assuming the radius equal to unit, expressions are hence easily derived for the tangents of multiple arcs. Let t denote the tangent of an arc a; then $1 - t^2 : 1 :: 2t : tan2a = \frac{2t}{1-t^2}$, and $1 - t.\frac{2t}{1-t^2} : 1 :: t + \frac{2t}{1-t^2} : tan3a = \frac{3t - t^3}{1-3t^2}$. In like manner, it will be found that,

$$Tan\ 4a = \frac{4t - 4t^3}{1 - 6t^2 + t^4}.$$

$$(7.)\quad Tan\ 5a = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}.$$

$$Tan\ 6a = \frac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6}.$$

&c. &c. &c. ‡.

* See Note LXVIII. † See Note LXIX.

‡ See Note LXX.

These *formulæ* might also be derived from expressions for the sine and cosine of the multiple arc which involve the powers of the tangent. Thus, from (1), $sin2a = 2cs = c^2\left(2\frac{s}{c}\right) = c^2.2t$, and $sin3a = 4c^2s - s = 3c^2s - (1-c^2)s = c^3\left(3\frac{s}{c} - \frac{s^3}{c^3}\right) = c^3(3t - t^3)$; again, from (2), $cos2a = 2c^2 - 1 = c^2 - s^2 = c^2\left(1 - \frac{s^2}{c^2}\right) = c^2(1-t^2)$, and $cos3a = 4c^3 - 3c = c^3 - 3c(1-c^2) = c^3\left(1 - 3\frac{s^2}{c^2}\right) = c^3(1-3t^2)$. In this way, the following tables are formed:

$$(8.)\quad \begin{aligned} Sin\ 2a &= c^2.2t. \\ Sin\ 3a &= c^3(3t - t^3). \\ Sin\ 4a &= c^4(4t - 4t^3). \\ Sin\ 5a &= c^5(5t - 10t^3 + t^5). \\ Sin\ 6a &= c^6(6t - 20t^3 + 6t^5). \end{aligned}$$

&c. &c. &c.

$$(9.)\quad \begin{aligned} Cos\ 2a &= c^2(1 - t^2). \\ Cos\ 3a &= c^3(1 - 3t^2). \\ Cos\ 4a &= c^4(1 - 6t^2 + t^4). \\ Cos\ 5a &= c^5(1 - 10t^2 + 5t^4). \\ Cos\ 6a &= c^6(1 - 15t^2 + 15t^4 - t^6). \end{aligned}$$

&c. &c. &c.

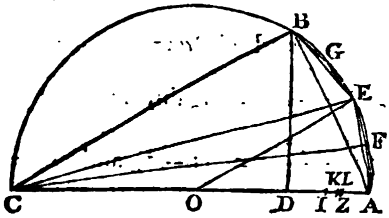
The first set of expressions being divided by the second, will evidently give the same results for the tangent of the multiple arc *.

PROP. VI. THEOR.

The supplemental chord of half an arc, is a mean proportional between the radius, and the sum of the diameter and the supplemental chord of the whole arc.

* See Note LXXI.

This property, which is only a modification of cor. 2. to Pr. 2. will admit of a more direct demonstration. For draw the chord AB, the semichords AE and BE, and the supplemental chords CB and CE, and the radius OE. The isosceles triangles AEB and COE are similar, for the angles OCE and EAB at the base stand on equal arcs AE and EB; consequently AE : AB : : CO : CE. But, ACBE being a quadrilateral figure contained in a circle, CE.AB=AE.CB+EB.CA=AE (CA+CB), or AE : AB : : CE : CA+CB; wherefore CO : CE : : CE : CA+CB, or $CE^2=CA\left(\frac{CA+CB}{2}\right)$.



Cor. Hence, in small arcs, the ratio of the sine to the arc approaches that of equality. For, let the semiarcs AE and EB be again bisected in the points F and G; and, continuing their subdivision indefinitely, let the successive intermediate chords be drawn. The ratio of the sine BD to the arc AB may be viewed as compounded of the ratio of BD to the chord AB, of that of AB to the two chords AE and EB, of that of AE and EB to the four chords AF, FE, EG, and GB, and so forth. But these ratios, it has been shown, are the same respectively as those of the supplemental chords CB, CE, CF, &c. to the diameter CA. And since each of the ratios CB : CA, CE : CA, CF : CA, &c. approaches to equality, it is evident that their compounded ratio, or that of the sine to its corresponding arc, must also approach to equality.

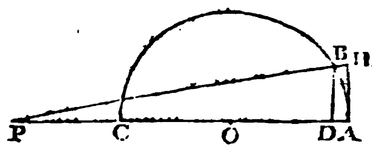
Scholium. Hence the ratio of the sine BD to the arc AB is expressed numerically, by the ratio of the continued product of the series of supplemental chords CB, CE, CF, &c. to the relative continued power of the diameter CA. The ratio may, therefore, be determined to any degree of exactness, by

the repeated application of the proposition in computing those derivative chords. But a very convenient approximation is more readily assigned. Make CD to CI as CB to CA, CI to CK as CE to CA, CK to CL as CF to CA, and so forth, tending always towards the limit Z; then the ratio of CD to CZ, being compounded of these ratios, must express the ratio of the sine BD to its corresponding arc AB. Now CD : CB : : CB : CA; consequently CI=CB, and CD : CI : : CI : CA, or the point I nearly bisects DA. Again, $CE^2 = CA\left(\frac{CA+CB}{2}\right)$, and therefore CE differs from CA, by only the fourth part of the difference between CB and CA. These differences being small in comparison of the quantities themselves, the series of supplemental chords may be considered as forming a regular progression, each succeeding term of which approaches four times nearer to the length of the diameter. Wherefore $IK = \frac{1}{4}DI$, $KL = \frac{1}{4}IK$, and so continually. But (V. 21. El.) as the difference between the first and second term, is to the first, so is the difference between the first and last term, or DI itself, to the sum of all the terms, or the extreme limit DZ; that is, 3 : 4 : : DI : DZ; and consequently $DZ = \frac{2}{3}DA$. The ratio of the sine BD to the arc AB is, therefore, nearly that of CD to $CD + \frac{2}{3}DA$, or of 3CD to CD+2CA.

This approximation may be differently modified. Since 3CD = 6OA—3DA, and CD + 2AC = 6OA—DA, it follows that BD is to AB, as 6OA—3DA to 6OA—DA. But this ratio, which approaches to equality, will not be sensibly affected, by annexing or taking away equal small differences. Whence the sine is to the arc, as 6OA—6DA to 6OA—4DA, or 3OD to OA+2OD. But OD is to OA, as the sine of AB is to its tangent; and consequently the triple of that arc is equal to its tangent together with twice its sine.

Again, both terms of the ratio increased by the minute difference DA become 6OA—2DA, and 6OA; wherefore

BD is to AB, as 3OA—DA to 3OA, or as 2OC+OD to 3CO. Hence, if CP be made equal to the radius CO, and PBH be drawn to meet the tangent, —the arc AB will be nearly equal to the intercepted portion AH. For BD : AH :: PD : PA, or 2OC+OD : 3OC; that is, as the sine BD is to its arc AB.



Another approximation, of much higher importance, may be hence derived; for PD : PA : : BD : AH, or as the sine to its arc nearly. But (V. 3. El.) PD × CD is to PA × CD in the same ratio, and PA × CD = PD × CD + AD × CD = (III. 32. cor. 1.) PD × CD + BD²; whence PD × CD is to PD × CD + BD², as the sine to its arc nearly. If the arc be small, it is evident that OD will be very nearly equal to AO, and consequently PD may be assumed equal to 3AO, and CD equal to 2AO. Wherefore 6AO² : 6AO² + BD² : : BD : AB nearly; or, the radius being unit, and a and s denoting a small arc and its sine, $6 : 6 + s^2 :: s : a$, and hence $a = s + \frac{s^3}{6}$ nearly. But since a and s are very small, a^3 will approach extremely near to s^3, and it may, therefore, be inferred conversely, that $s = a - \frac{a^3}{6}$.

A convenient approximation for the versed sine of an arc is easily derived from the fundamental property of the lines themselves; for 2AO.AD = AB² = BD² + AD², or employing v to denote the versed sine, $2v = s^2 + v^2$, and $v = \frac{s^2}{2} + \frac{v^2}{2}$. If, therefore, the arc be small, it may be sufficiently near the truth to assume $v = \frac{s^2}{2}$; but should greater accuracy be required, substitute this value of v in the second term of the complete expression, and $v = \frac{s^2}{2} + \frac{s^4}{8}$, which will form a very close approximation.

Calculation of the Trigonometrical Lines.

The preceding theorems contain all the principles required in constructing Trigonometrical Tables. The radius being denoted by unit, the several lines connected with the circle are referred to that standard, and are generally computed to seven decimal places.

The first object is to compute the SINES for every arc of the quadrant.

Since the semicircumference of a circle whose radius is unit was found, by the scholium to Prop. 32. Book VI. of the Elements, to be 3.1415926, the length of the arc of one minute is .0002909, which, in so small an arc, may be assumed as equal to the sine, and consequently the versed sine of a minute $= \frac{1}{2}(.0002909)^2 = .000{,}000{,}042{,}308$. Whence, by cor. 3. to Prop. 3. $sin(A+1') = 2sinA - 2sinA \times .000{,}000{,}042{,}308 - sin\,(A-1')$; and therefore, by a series of repeated operations, the intermediate arc being successively 1′, 2′, 3′, 4′, &c. the sines of 2′, 3′, 4′, 5′, &c. in their order will be calculated.

The numbers thus obtained will at first scarcely differ from an uniform progression, the versed sine of 1′, which forms the multiplier of deviation, being so extremely small. It is hence superfluous to compute rigidly all those minute variations. The labour may be greatly shortened, by calculating the sines for each degree only, and employing some abridged process for filling up the sines, corresponding to the subdivision in minutes.

The arc of one degree being equal to .0174533, it follows from the scholium to Prop. 6., that the sine of $1° = .0174533 - \frac{1}{6}(.0174533)^3 = .0174524$, and hence the

versed sine of $1° = \frac{1}{2}(.0174524)^2 = .0001523$. Wherefore $sin(A+1°) = 2sinA - 2sinA \times .0001523 - sin(A-1°)$; or, *if from twice the sine of an arc, diminished by its 6566th part, the sine of an arc one degree lower be subtracted, the remainder will exhibit the sine of an arc, which is one degree higher.* Thus,

$Sin 2° = 2 sin 1 - 2 sin 1° \times .0001523 = .0349048 - .0000053$
$= .0348995$

$Sin 3° = 2 sin 2° - 2 sin 2° \times .0001523 - sin 1° = .0697990 - .0000106 -$
$.0174524 = .0523360.$

$Sin 4° = 2 sin 3° - 2 sin 3° \times .0001523 - sin 2° = .1046720 - .0000160 -$
$.0348995 = .0697565.$

After this manner, the sine for each degree is computed in succession.

But the sines may be found, independently of the previous quadrature of the circle. Assuming an arc whose chord is already known, it is easy, from Prop. 6. to determine the successive chords and supplemental chords of its continued bisection. Let that arc be 60°; its chord is *equal* to the radius, and (IV. 20. cor. 2.) its supplemental chord $= \sqrt{3} = 1.7320508076$. Whence the supplemental chord of $30° = \sqrt{(2+1.7320508076)} = 1.9318516525$. In this way, by continued extractions, the supplemental chords of 15°, 7° 30′, 3° 45′, and 1° 52′½ are successively computed, the last one being equal to 1.9997322758. Again, the chords themselves are deduced by a series of analogies; for 1.9318516525 : 1 : : 1 : .5176380900 4 = chord of 15°, and so repeatedly, till the chord of 1° 52′½, which is .0327234633. Hence, taking the halves of those numbers, the sine of $56'\frac{1}{4} = .0163617317$ and the cosine of $56'\frac{1}{4} = 9998661379$, and therefore (cor. 3. defin.) the tangent of that arc is .0163639215; consequently the arc itself $\frac{1}{3}$ (2 × .0163617317 + .0163639215) = .0163624616, and thence the length of the arc of a minute is .0002908882086.

Wherefore the sine of $1' = .0002908882 - \frac{1}{6}(.0002908882)^3 = .00029088826046$, and the versed sine of $1' = \frac{1}{2}(.00029088826046)^2 = .000000042308$.

Employing these data, therefore,

$Sin 2' = 2 sin 1' - 2 sin 1' \times .000000042308 = .0005817763845$;

$Sin 3' = 2 sin 2' - 2 sin 2' \times .000000042308 - sin 1' = .0008726645152$;

and so forth *.

But it is very seldom requisite to push the estimation to such extreme nicety. The sines being calculated for each degree as before, those corresponding to the subdivision in minutes, may be found by a more expeditious method, though founded on ulterior considerations. If the sines increased uniformly, the sine of $A° + n'$ would exceed that of A by the quantity $\frac{n}{120}(sin \overline{A + 1°} - sin \overline{A - 1°}) = B$. But the rate of this augmentation, being continually retarded, occasions a defect, equal to $n^2 \times sin A \times .000,000,042308 = C$. Again, since the retardation itself gradually relaxes, it requires a small compensation, which may be estimated at $(60 - n)B \times .0000013 = D$. The sine of $A° + n'$ is then very nearly $= sin A + B - C + D$. Thus, the sines of 31°, 32°, and 33° being respectively .5150381, .5299193, and .5446390, let it be required to find the sine of 32° 40′. Here $B = \frac{40}{120}(sin 33° - sin 31°) = .0098670$, $C = 1600 \times sin 32° \times .0000000423 = .0000359$, and $D = 20 \times .0098670 \times .0000013 = .0000003$. Whence $sin 32° \; 40' = .5299193 + .0098670 - .0000359 + .0000003 = .5397507$.

After the sines are calculated up to 60°, the rest are deduced from cor. 4. Prop. 3. by simple addition. Thus, $sin 61° = sin 59° + sin 1° = .8571673 + .0174524 = .8746197$ †.

The VERSED SINES and supplementary versed sines are only the difference and sum of the radius and the sines.

---

* See Note LXXII. † See Note LXXIII.

The TANGENTS are easily derived from the sines, by help of the analogy given in the 3d corollary to the definitions. Thus, $\cos 32° : \sin 32° :: R : \tan 32°$, or, $.8480481 : .5299193 :: 1 : .6248694 = \tan 32°$. Beyond 45°, the calculation is simplified, the radius being (cor. 7. defin.) a mean proportional between the tangent and cotangent, or the cotangent is the reciprocal of the tangent.

The SECANTS are deduced by cor. 4. to the definitions, since they are the reciprocals of the cosines.

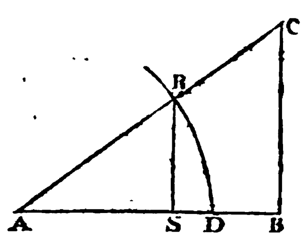
From the lower tangents and secants, the tangents of arcs that exceed 45° are most easily derived; for (cor. 4. Prop. 4.) $\tan(45° + a) = \sec 2a + \tan 2a$. Thus, $\tan 46° = \sec 2° + \tan 2°$, or $1.0355303 = 1.0006095 + .0349208$.

PROP. VII. THEOR.

In a right angled triangle, the radius is to the sine of an oblique angle, as the hypotenuse to the opposite side.

Let the triangle ABC be right angled at B; then $R : \sin CAB :: AC : CB$.

For assume AR equal to the given radius, describe the arc RD, and draw the perpendicular RS. The triangles ARS and ACB are evidently similar, and therefore $AR : RS :: AC : CB$. But, AR being the radius, RS is the sine of the arc RD which measures the angle RAD or CAB; and consequently $R : \sin A :: AC : CB$.



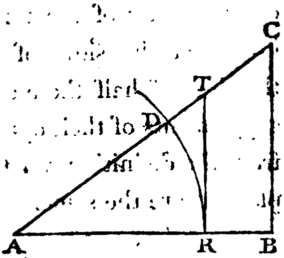
Cor. Hence the radius is to the cosine of an angle, as the hypotenuse to the adjacent side; for $R : \sin C$ or $\cos A :: AC : AB$.

PROP. VIII. THEOR.

In a right angled triangle, the radius is to the tangent of an oblique angle, as the adjacent side to the opposite side.

Let the triangle ABC be right angled at B; then $R : tan$BAC : : AB : BC.

For, assuming AR equal to the given radius, describe the arc RD, and draw the perpendicular RT. The triangles ART and ABC being similar, AR : RT : : AB : BC. But, AR being the radius, RT is the tangent of the arc RD which measures the angle at A; and therefore $R : tan$A : : AB : BC.



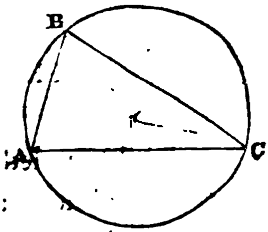
Cor. Hence the radius is to the secant of an angle, as the adjacent side to the hypotenuse. For AT is the secant of the arc RD, or of the angle at A; and, from similar triangles, AR : AT : : AB : AC.

PROP. IX. THEOR.

The sides of any triangle are as the sines of their opposite angles.

In the triangle ABC, the side AB is to BC, as the sine of the angle at C to the sine of that at A.

For let a circle be described about the triangle; and the sides AB and BC, being chords of the intercepted arcs



or of the angles at the centre, are (cor. def.) equal to twice the sines of the halves of those angles, or the angles ACB and CAB at the circumference. But, of the same angles, the chords or sines (VI. 12, cor. El.) are proportional to the radius; and consequently AB : BC : : *sin*C : *sin*A.

Cor. Since the straight lines AB and BC are chords, not only of the arcs AB and BC, but of the arcs ACB and BAC, or the defects of the former from the circumference, it follows that the sides of the triangle are proportional also to the sines of half these compound arcs, or to the sines of the supplements of their opposite angles.—A like inference results from the definition, for the sine of an arc and that of its supplement are the same.

PROP. X. THEOR.

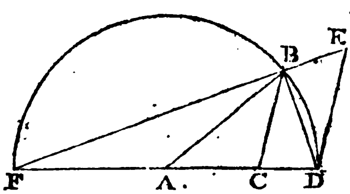
In any triangle, the sum of two sides, is to the difference, as the tangent of half the sum of the angles at the base, to the tangent of half their difference.

In the triangle ABC, AB+AC : AB—AC : : $tan\frac{C+B}{2}$: $tan\frac{C-B}{2}$.

For, by the last proposition, AB : AC : : *sin*C : *sin*B, and consequently (V. 12. El.) AB+AC : AB—AC : : *sin*C+*sin*B : *sin*C—*sin*B. But, by Prop. 4. *sin*C+*sin*B : *sin*C—*sin*B : : $tan\frac{C+B}{2}$: $tan\frac{C-B}{2}$; wherefore, by identity of ratios, AB+AC : AB—AC : : $tan\frac{C+B}{2}$: $tan\frac{C-B}{2}$.

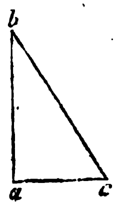
Otherwise thus.

From the vertex A, and with a distance equal to the greater side AB, describe the semicircle FBD, meeting the other side AC extended both ways to F and D, join BD and BF, which produce to meet a straight line DE drawn parallel to CB.



Because the isosceles triangle DAB, has the same vertical angle with the triangle CAB, each of its remaining angles ADB and ABD is (I. 32. El.) equal to half the sum of the angles ACB and ABC; and therefore the defect of ABC from that mean, that is the angle CBD, or its alternate angle BDE, must be equal to half the difference of those angles. Now FBD being (III. 22. El.) a right angle, BF and BE are tangents of the angles BDF and BDE, to the radius DB, and hence are proportional to the tangents of those angles with any other radius. But since CB and DE are parallel, CF, or AB+AC : CD, or AB—AC : : BF : BE; consequently AB+AC : AB—AC $::\tan\frac{ACB+ABC}{2}:\tan\frac{ACB-ABC}{2}$, or AB+AC : AB—AC $::\cot\frac{1}{2}A:\cot(B+\frac{1}{2}A)$, or $-\cot(C+\frac{1}{2}A)$.

Cor. Suppose another triangle *abc* to have the sides *ab* and *ac* equal to AB and AC, but containing a right angle: It is obvious that $\tan\frac{c+b}{2}:\tan\frac{c-b}{2}$ $::\tan\frac{ACB+ABC}{2}:\tan\frac{ACB-ABC}{2}$, or



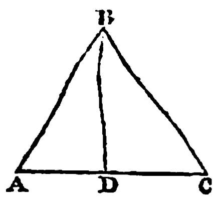
$R:\tan(45^\circ-b)::\tan\frac{ACB+ABC}{2}:\tan\frac{ACB-ABC}{2}$,

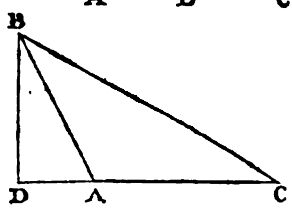
that is, $R:\tan(45-b)::\cot\frac{1}{2}A:\cot(B+\frac{1}{2}A)$, or $-\cot(C+\frac{1}{2}A)$. Now, in the right angled triangle *abc*, *ab* or AB, is to *ac*, or AC, as the radius, to the tangent of the angle at *b*.

PROP. XI. THEOR.

In any triangle, as twice the rectangle under two sides, is to the difference between their squares and the square of the base, so is the radius, to the cosine of the contained angle.

In the triangle ABC, $2AB \times AC : AB^2 + AC^2 - BC^2 :: R : cos BAC$; the angle BAC being acute or obtuse, according as BC^2 is less or greater than $AB^2 + AC^2$.





For let fall the perpendicular BD. In the right angled triangle ADB, $AB : AD :: R : sin ABD$ or $cos BAC$; consequently $2AB \times AC : 2AD \times AC :: R : cos BAC$. But (II. 26. El.) twice the rectangle under AD and AC is equal to the difference of the squares AB and AC from the square of BC. Whence $2AB \times AC : AB^2 + AC^2 - BC^2 :: R : cos\ BAC$.

Cor. The radius being denoted by unit, it follows (V. 6. El.) that $AB^2 + AC^2 - BC^2 = 2AB \times AC \times cos BAC$, and consequently $BC^2 = AB^2 + AC^2 - 2AB \times AC \times cos BAC$, or $BC = \sqrt{(AB^2 + AC^2 - 2AB \times AC \times cos BAC)}$.

PROP. XII. THEOR.

In any triangle, the rectangle under the semiperimeter and its excess above the base, is to the rectangle under its excesses above the two sides, as the square of the radius, to the square of the tangent of half the contained angle.

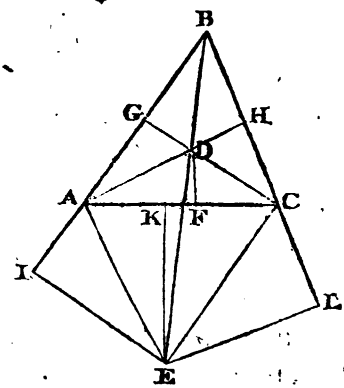
In the triangle ABC, the perimeter being denoted by P, $\frac{1}{2}P(\frac{1}{2}P - AC) : (\frac{1}{2}P - AB)(\frac{1}{2}P - BC) :: R^2 : tan \frac{1}{2}B^2$.

For, employing the same construction as in Prop. 31., Book VI. of the Elements; since the triangles BIE and BGD are right angled, BI : IE : : R : *tan*IBE, or *tan*$\frac{1}{2}$B, and

BG : GD : : R : *tan*GBD, or *tan*$\frac{1}{2}$B; whence

(V. 22. El.) BI × BG : IE × GD : : R^2 : *tan*$\frac{1}{2}B^2$.

But it was proved that IE × GD = AI × AG; wherefore BI × BG : AI × AG :: R^2 : *tan*$\frac{1}{2}B^2$. Now BI is equal to the semiperimeter, BG is its excess above the base AC, and AI, AG are its excesses above the sides AB and BC; consequently the proportion is established.



PROP. XIII. THEOR.

In any triangle, the rectangle under two sides, is to the rectangle under the semiperimeter, and its excess above the base, as the square of the radius, to the square of the cosine of half the contained angle.

In the triangle ABC, the perimeter being denoted by P, AB × BC : $\frac{1}{2}$P($\frac{1}{2}$P—AC) : : R^2 : *cos*$\frac{1}{2}B^2$.

For, the same construction remaining; in the right angled triangles BIE and BGD,

BE : BI : : R : *sin*BEI, or *cos*$\frac{1}{2}$B,

and BD : BG : : R : *sin*BDG, or *cos*$\frac{1}{2}$B;

whence BE × BD : BI × BG : : R^2 : *cos*$\frac{1}{2}B^2$.

But the quadrilateral figure EADC being right angled at A and C, is (III. 19. cor.) contained in a circle, and consequently (III. 18. El.) the angle AED or AEB is equal to

ACD or to DCB; wherefore, since by construction the angle ABE is equal to DBC, the triangles BAE and BDC are similar, and BE : AB :: BC : BD, or $BE \times BD = AB \times BC$. Hence $AB \times BC : BI \times BG :: R^2 : cos\frac{1}{2}B^2$. The proposition is therefore demonstrated.

PROP. XIV. THEOR.

In any triangle, as the rectangle under two sides is to the rectangle under the excesses of the semiperimeter above those sides, so is the square of the radius, to the square of the sine of half their contained angle.

In the triangle ABC, the perimeter being still denoted by P, $AB \times BC : (\frac{1}{2}P - AB)(\frac{1}{2}P - BC) :: R^2 : sin\frac{1}{2}B^2$.

For, the same construction being retained, in the right angled triangles BIE and BGD, $BE : IE :: R : sin\frac{1}{2}B$,

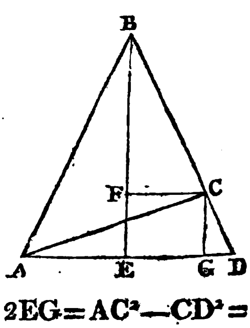
and $BD : GD :: R : sin\frac{1}{2}B$;

whence $BE \times BD : IE \times GD :: R^2 : sin\frac{1}{2}B^2$.

But it has been proved that $BE \times BD = AB \times BC$, and $IE \times GD = AI \times AG$, or the rectangle under the excesses of the semiperimeter above the sides AB and BC; wherefore the proposition is established.

Otherwise thus:

Produce the shorter side BC till BD be equal to AB, join AD, let BE bisect the vertical angle, and draw CG and CF parallel to BE and AD. Since BE is perpendicular to ED and FC, it follows that BD, or $AB : ED :: R : sin\frac{1}{2}B$, and $BC : FC$, or $EG :: R : sin\frac{1}{2}B$. Wherefore $AB \times BC : ED \times EG :: R^2 : sin\frac{1}{2}B^2$.



Now (II. 24. El.) $2ED \times 2EG = AC^2 - CD^2 =$ (II. 19. El.) $(AC + CD)(AC - CD)$, and consequently

$ED \times EG = \left(\frac{AC+CD}{2}\right)\left(\frac{AC-CD}{2}\right)$; but $\frac{AC+CD}{2} = \frac{AC+AB-BC}{2} = \frac{P-2BC}{2} = \frac{1}{2}P-BC$, and $\frac{AC-CD}{2} = \frac{AC-(AB-BC)}{2} = \frac{P-2AB}{2} = \frac{1}{2}P-AB$. Hence, by substitution, $AB \times BC : (\frac{1}{2}P-AB)(\frac{1}{2}P-BC) :: R^2 : sin\frac{1}{2}B^2$.

The eight preceding theorems contain the grounds of trigonometrical calculation. A triangle has only five variable parts,—the three sides and two angles, the remaining angle being merely supplemental. Now, it is a general principle, that, three of those parts being given, the rest may be thence determined. But the right angled triangle has necessarily one known angle; and, in consequence of this, the opposite side is deducible from the containing sides. In right angled triangles, therefore, the number of parts is reduced to four, any two of which being the assigned, the others may be found.

PROP. XV. PROB.

Two variable parts of a right angled triangle being given, to find the rest.

This problem divides itself into four distinct cases, according to the different combination of the data.

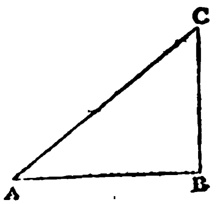
1. *When the hypotenuse and a side are given.*

2. *When the two sides containing the right angle are given.*

3. *When the hypotenuse and an angle are given.*

4. *When either of the sides and an angle are given.*

The first and third cases are solved by the application of Proposition 7, and the second and fourth cases receive their solution from Proposition 8. It may be proper, however, to exhibit the several analogies in a tabular form.



Case.	Given.	Sought.	SOLUTION.
I	AC, AB	A, or C, BC	AC : AB : : *R* : *sin*C, or *cos*A. *R* : *sin*A : : AC : BC.
II	AB, BC	A, or C AC.	AB : BC : : *R* : *tan*A, or *cot* C. *cos*A : *R* : : AB : AC, or *R* : *sec*A : : AB : AC
III	AC A	AB BC	*R* : *cos*A : : AC : AB. *R* : *sin*A : : AC : BC.
IV	AB, A	BC AC	*R* : *tan*A : : AB : BC. *cos*A : *R* : : AB : AC, or *R* : *sec*A : : AB : AC.

In the first and second cases, BC or AC might also be deduced, by the mere application of Prop. 11. Book II. of the Elements: For $AC^2 = AB^2 + BC^2$, or $AC = \sqrt{(AB^2+BC^2)}$ and $BC^2 = AC^2 - AB^2 = (AC + AB)(AC - AB)$, or $BC = \sqrt{((AC+AB)(AC-AB))}$.

Cor. Hence the first case admits of a simple approximation. For, by the scholium to Proposition 6, it appears, that, AC being made the radius, 2AC+AB is to 3AC, as the side BC is to the arc which measures its opposite

angle CAB, or alternately 2AC+AB is to BC, as 3AC to the arc corresponding to BC. But the radius is equal to an arc of $57° 17' 44'' 48'''$, or $57\frac{1}{7}°$ nearly; wherefore 3AC is to the arc which corresponds to BC, as $3 \times 57\frac{1}{7}°$, or 172°, to the number of degrees contained in the angle CAB, and consequently 2AC+AB : BC : : 172° : the expression of the angle at A, or AC+½AB : BC : : 86° : number of degrees in the angle at A.

This approximation will be the more correct, when the side opposite to the required angle becomes small in comparison with the hypotenuse; but the quantity of error can never amount to 4 minutes.

PROP. XVI. PROB.

Three variable parts of an oblique angled triangle being given, to find the other two.

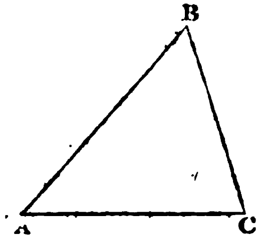
This general problem includes three distinct cases, one of which again is branched into two subordinate divisions.

1. *When all the three sides are given.*

2. *When two sides and an angle are given; which angle may either* (1.) *be contained by these sides, or* (2.) *subtended by one of them.*

3. *When a side and two of the angles are given.*

The first case admits of four different solutions, derived from Propositions 11, 12, 13, and 14, and which have their several advantages. The second case, consisting of two branches, is resolved by the application of propositions 9 and 10; and the solution of the third case flows immediately from the former of these propositions.



Case.		Given.	Sought.	SOLUTION.
I.		AB, BC, and AC.	B.	$AB \times BC : (\frac{1}{2}P - AB)(\frac{1}{2}P - BC) :: R^2 : \sin\frac{1}{2}B^2$ $\frac{1}{2}P(\frac{1}{2}P - AC) : (\frac{1}{2}P - AB)(\frac{1}{2}P - BC) :: R^2 : \tan\frac{1}{2}B^2$ $AB \times BC : \frac{1}{2}P(\frac{1}{2}P - AC) :: R^2 : \cos\frac{1}{2}B^2$ $2AB \times BC : AB^2 + BC^2 - AC^2 :: R : \cos\frac{1}{2}B$
II.	1	AB, BC, and C.	A, and AC.	$AB : BC :: \sin C : \sin A$; whence B, and $\sin C : \sin B :: AB : AC$.
	2	AB, BC, and B.	A, or C, and AC.	$AB + BC : AB - BC :: \cot\frac{1}{2}B :: \cot(A + \frac{1}{2}B)$, or $-\cot(C + \frac{1}{2}B)$. $\begin{cases} AB : BC :: R :: \tan b; \text{ and} \\ R : \tan(45^\circ - b) :: \cot\frac{1}{2}B : \cot(A + \frac{1}{2}B,) \end{cases}$ or $-\cot(C + \frac{1}{2}B.)$ $\sin A : \sin B :: BC : AC$, or $AC = \sqrt{(AB^2 + BC^2 - 2AB \times BC \times \cos B.)}$
III.		AB, A, B, and thence C.	BC AC	$\sin C : \sin A :: AB : BC$. $\sin C : \sin B :: AB : AC$.

For the resolution of the first Case, the analogy set down first, is on the whole the most convenient, particularly if the angle sought do not approach to two right angles. The second analogy may be applied through a wider extent, but is liable in practice to some irregularity, when the angle sought becomes very obtuse. The third and fourth analogies, especially the latter, are not adapted for the calculation of very

acute angles; they will, however, answer the best when the angle sought is obtuse. It is to be observed, that the cosines of an angle and of its supplement are the same, only placed in opposite directions; and hence the second term of the analogy, or the difference of AB^2+BC^2 from AC^2, is in excess or defect, according as the angle at B is acute or obtuse.— These remarks are founded on the unequal variation of the sine and tangent, corresponding to the uniform increase of an arc *.

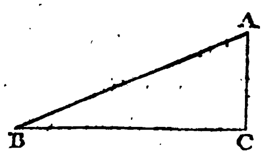
The first part of Case II. is ambiguous, for an arc and its supplement have the same sine. This ambiguity, however, is removed if the character of the triangle, as acute or obtuse, be previously known.

For the solution of the second part of Case II. the first analogy is the most usual, but the double analogy is the best adapted for logarithms. The direct expression for the side subtending the given angle is very commodious, where logarithms are not employed †.

PROP. XVII. PROB.

Given the horizontal distance of an object and its angle of elevation, to find its height and absolute distance.

Let the angle CAB, which an object A makes at the station B, with an horizontal line, and also the distance BC of a perpendicular AC, to find that perpendicular, and the hypotenusal or aerial distance BA.



In the right angled triangle BCA, the radius is to the tangent of the angle at B, as BC to AC; and the radius is to the secant of the

* See Note LXXIV. † See Note LXXV.

angle at B, or the cosine of the angle at B is to the radius, as BC to AB.

PROP. XVIII. PROB.

Given the acclivity of a line, to find its corresponding vertical and horizontal length.

In the preceding figure, the angle CBA and the hypotenusal distance BA being given to find the height and the horizontal distance of the extremity A.

The triangle BCA being right angled, the radius is to the sine of the angle CBA as BA to AC, and the radius is to the cosine of CBA as BA to BC.

Scholium. If the acclivity be small, and A denote the measure of that angle in minutes; then $AC = BA \times \frac{A}{3438}$ nearly. But the expression for AC, will be rendered more accurate, by subtracting from it, as thus found, the quantity $\frac{AC^3}{BA^2}$.

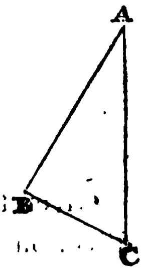
In most cases when CBA is a small angle, the horizontal distance may be computed with sufficient exactness, by deducting $\frac{AC^2}{2BA}$, or $BA \times A^2 \times .000,000,0423$, from the hypotenusal distance.

PROP. XIX. PROB.

Given the interval between two stations, and the direction of an object viewed from them, to find its distance from each.

Let BC be given, with the angles ABC and ACB, to calculate AB and AC.

In the triangle CBA, the angles ABC and ACB being given, the remaining or supplemental angle BAC is thence given; and consequently, $\sin BAC : \sin ACB :: BC : AB$, and $\sin BAC :: \sin ABC : BC : AC$.

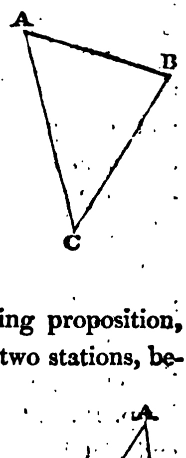


Cor. If the observed angles ABC and ACB be each of them 60°, the triangle will be evidently equilateral; and if the angle at the station B be 90° and that at C 45°, the distance AB will be equal to the base BC.

PROP. XX. PROB.

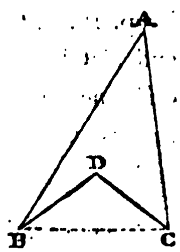
Given the distances of two objects from any station and the angle which they subtend, to find their mutual distance.

Let AC, BC, and the angle ACB be given, to determine AB.



In the triangle ABC, since two sides and their contained angle are given, therefore, by cor. Prop. 10. $AC + BC : AC - BC :: \cot\frac{1}{2}C : \cot(A + \frac{1}{2}C)$, and then $\sin A : \sin C :: BC : AB$; or (by corollary to Prop. 11.) $AB = \sqrt{(AC^2 + BC^2 - 2AC.BC \cos C)}$.

Cor. By combining this with the preceding proposition, the distance of an object may be found from two stations, between which the communication is interrupted. Thus let A be visible from B and C, though the straight line BC cannot be traced. Assume a third station D, from which B and C are both seen. Measure DB and DC, and observe the angles BDC, ABC and ACB. In the triangle BDC, the base BC is found as above; and thence, by the preceding proposition, the sides AB and AC of the triangle ABC are determined.



PROP. XXI. PROB.

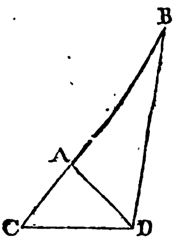
Given the interval between two stations, and the directions of two remote objects viewed from them in the same plane, to find the mutual distance, and relative position of those objects.

Let the points A, B represent the two objects, and C, D the two stations from which these are observed; the interval or base CD being measured, and also the angles CDA, CDB at the first station, and DCA, DCB at the second; it is thence required to determine the transverse distance AB, and its direction.

It is obvious that each of the points A and B would be assigned geometrically by the intersection of two straight lines, and consequently that the position of the objects will *not* be determined, unless each of them appears in a different direction at the successive stations.

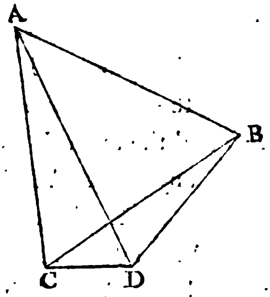
1. *Suppose one of the stations C to lie in the direction of the two objects A and B.*

At C observe the angle BCD, and at D the angles CDA and BDC. Then by Prop. 19. *sin*CAD : *sin*CDA :: CD : CA, and *sin*CBD : *sin*CDB :: CD : CB; the difference or sum of CA and CB is AB, the distance sought.



2. *When neither station lies in the direction of the two objects, and the base CD has a transverse position.*

Find by Prop. 19. the distances AC and BC of both ob-



jects from one of the stations C; then the contained angle ACB, or the excess of DCA above DCB, being likewise given, the angles at the base AB of the triangle BCA, and the base itself, may be calculated, from the analogies exhibited for the solution of the second branch of Case II. For $AC+BC : AC-BC :: \cot\frac{1}{2}ACB : \cot(\frac{1}{2}ACB+CAB)$, and thus the angle CAB is found. Or more conveniently by two successive operations, $AC : BC :: R : \tan b$, and $R : \tan(45^\circ - b) :: \cot\frac{1}{2}ACB :: \cot(\frac{1}{2}ACB+CAB.)$ Now, $\sin CAB :: \sin ACB : BC : AB$, or $AB = \sqrt{(AC^2+BC^2-2AC \times BC \times \cos ACB)}$.

The inclination of AB to CD in the first case is given by observation, and in the second case it is evidently the supplement of the interior angles CAB and DCA. A parallel to AB may hence be drawn from either station.

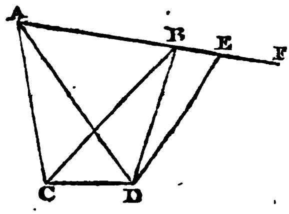
Cor. Hence the converse of this problem is readily solved. Suppose two remote objects A and B, of which the mutual distance is already known, are observed from the stations C and D, and it were thence required to determine the interval CD. Assume unit to denote CD, and calculate AB according to the same scale of measures; the actual distance AB being then divided by that result, will give CD: For the several triangles which combine to form the quadrilateral figure CABD, are evidently given in species.

PROP. XXII. PROB.

Given the directions of two inaccessible objects viewed in the same plane from two given stations, to trace the extension of the straight line connecting them.

Let the angles ACD, BCD be observed at C, and ADC, BDC at D, with the base CD; to find a point E in the straight line ABF produced through A and B.

By the last proposition, find AD and the angle DAB, and assume any angle ADE. In the triangle DAE, the angles at the base AD, and consequently the vertical angle AED, being known, it follows, by Prop. 9., that

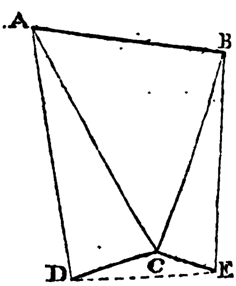


*sin*AED : *sin*EAD : : AD : DE. Measure out DE, therefore, on the ground, and its extremity E will mark the extension of AB.

PROP. XXII. PROB.

Given on the same plane the direction of two remote objects separately seen from two stations, and their direction as viewed at once from an intermediate station, with the distances of those stations, from the middle station,—to find the mutual distance of the objects.

Let object A be visible from the station D, and B from E, and both of them be seen at once from the station C; the compound base DC, CE being measured, and the angles DCA, ACB and BCE, with ADC and BEC, observed—to determine AB.



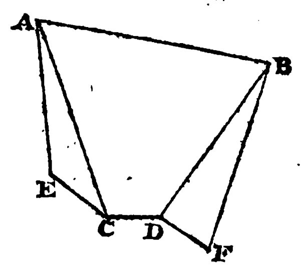
In the triangles DAC, CBE, the sides AC and BC are found by Prop. 19. and in the triangle ACB, the base AB is thence found by the application of Prop. 20.

It is evident that the mode of investigation will not be altered if the three stations D, C and E should lie in the same straight line.

PROP. XXIV. PROB.

Given four stations, with the direction of a remote object viewed from the first and second stations, and the direction of another remote object viewed from the third and fourth stations, all in the same plane,—to find the distance between the objects.

Let the bases EC, CD, and DF be given, with the angles ECD and CDF, and suppose that at the stations E and C the angles CEA and ECA are observed, and the angles BDF and BFD at D and F; to find the transverse distance AB.



In the triangles EAC and DBF, find by Prop. 19. the sides AC and BD; and, in the triangle CAD, the sides AC, CD, with their contained angle ACD, being given, the base DA and the angle CDA are found by Case II. But the distances DA, DB being now given, with their contained angle ADB, the base AB is found by Prop. 20.

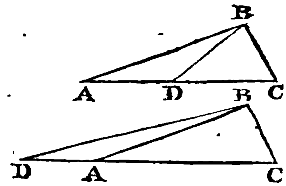
PROP. XXV. PROB.

The mutual distances of three remote objects being given, with the angles which they subtend at a station in the same plane, to find the relative place of that station.

Let the three points A, B, and C, and the angles ADB and BDC which they form at a fourth point D, be given; to determine the position of that point.

1. *Suppose the station D to be situate in the direction of two of the objects A, C.*

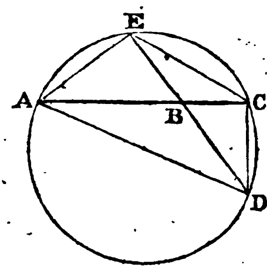
All the sides AB, AC and BC of the triangle ABC being given, the angle BAC is found by Case I.; and in the triangle ABD, the side AB with the angles at A and D being given, the side AD is found by Case III. and consequently the position of the point D is determined.



2. *Suppose the three objects A, B and C to lie in the same direction.*

Describe a circle about the extreme objects A, C and the station D, join DA, DB and DC, produce DB to meet the circumference in E, and join AE and CE.

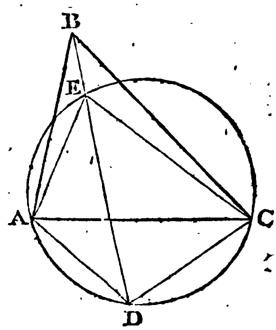
In the triangle AEC, the side AC is given, and the angles EAC and ECA, being equal (III. 18. El.) to CDE and ADE, are consequently given; wherefore the side AE is found by Case III. The triangle AEB, having thus the sides AE, AB, and their contained angle EAB or BDC given, the angle ABE and its supplement ABD are found by Case II. Lastly, in the triangle ABD, the angles ABD and ADB, with the side AB, are given; whence BD is found by Case III. But since the angle ABD and the distance BD are assigned, the position of the station D is evidently determined.



3. *Let the three objects form a triangle, and the station D lie either without or within it.*

Through D and the extreme points A and C describe a circle, draw DB cutting the circumference in E, and join AE and CE.

1. In the triangle AEC, the side AC, and the angles ACE and CAE, which are equal (III. 18. El.) to ADB and BDC, being given, the side AE is found by Case III.



2. All the sides of the triangle ABC being given, the angle CAB is found by Case I.

3. In the triangle BAE, the sides AB and AE are given, and their contained angle EAB, or the difference of CAE and CAB, are given, whence, by Case II., the angle ABE or ABD is found.

4. Lastly, in the triangle DAB, the side AB and the angles ABD and ADB being given, the side AD or BD is found by Case III., and consequently the position of the point D, with respect to A and B is determined. By a like process, the relative position of D and C is deduced; or CD may be calculated by Case II. from the sides AC, AD, and the angle ADC, which are given in the triangle CAD.

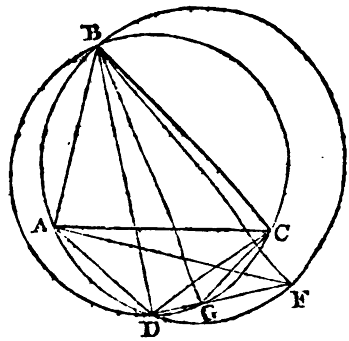
It is obvious that the calculation will fail, if the points B and E should happen to coincide. In fact, the circle then passing through B, any point D whatever in the opposite arc ADC will answer the conditions required, since the angles ADB, and BDC, being now in the same segment, must remain unaltered.

Otherwise thus.

On AB describe (III. 27. El.) a segment of a circle ADB containing an angle equal to that subtended by the objects A and B, and on BC describe another segment BDC containing an angle equal to that subtended by the objects B and C;

the point D, where the two circumferences intersect, will evidently mark the station required.

Join AD, BD, CD, draw the diameters BF, BG, and join AF, CG, DF and DG. The angles BDF and BDG, thus occupying semicircles, are right angles, and therefore DGF forms but one straight line. Hence these successive calculations.



1. All the sides of the triangle BAC being given, the angle ABC is found by Case I.

2. In the right-angled triangles BAF, BCG, the sides AB, BC, and the angles AFB, BGC, which are equal (III. 18. El.) to ADB, BDC, being given, the hypotenuses BF, BG, or the diameters of the circles are thence found.

3. In the triangle FBG the two sides BF, BG being now given, with the angle FBG=CBG—CBF=CBG—ABC+ABF=BAC+BCA—ADC, the angle BFG is found by Case II.

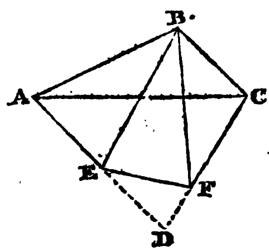
4. Lastly, in the right angled triangle BDF, the hypotenuse BF, and the angle BFD or BFG being given, the side BD is found; and since the angle FBD is also known, the position of the point D is assigned.

Should the two circles have the same centre, their circumferences must obviously coincide, and therefore every point in the containing arc will answer the conditions required. When this porismatic or indeterminate case of the problem occurs, the distances AB and BC become chords of the corresponding observed angles, and are consequently, by Proposition IX. proportional to the sines of those angles *.

* See Note LXXVI.

PROP. XXVI. PROB.

The mutual distances of three remote objects, two of which only are seen at once from the same station, being given, with the angles observed at two stations in the same plane, and the intermediate direction of these stations,—to find their relative places.



Suppose the three points A, B and C are given, with the angle AEB which A and B subtend at E, and BFC, which B and C subtend at F, and likewise the angles AEF and EFC; to find the relative situation of each of those stations E and F.

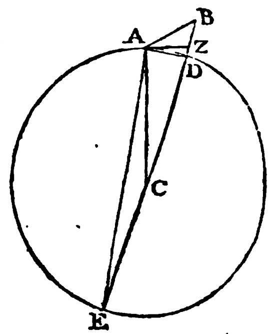
Produce AE and CF to meet in D, and join BD. The angle EDF, being equal to AEF + CFE — 180°, is given. Now in the triangle EBF, *sin*BFE : *sin*EBF : : EB : EF; and in the triangle EDF, *sin*EDF : *sin*DFE : : EF : ED; wherefore, (V. 23. El.) *sin*BFE × *sin*EDF : *sin*EBF × *sin*DFE : : EB : ED, and consequently the ratio of EB to ED is found. Again the angle BED, being the supplement of AEB, is given, (Prop. 10. cor.) *sin*BFE × *sin*EDF : *sin*EBF × *sin*DFE : : R : *tan* b, and R : *tan*$(45°-b)$: : *cot*$\frac{1}{2}$BED : −*cot*($\frac{1}{2}$BED + EBD), or *cot*(180° — $\frac{1}{2}$BED — EBD), whence the angle EDB is given. The angles which all the three objects A, B, and C subtend at the point D are therefore all given, and hence the position of D is determined by the preceding proposition. But BD, being found, the several distances BE, ED, and BF, FD are thence obtained, and consequently the position of each of the stations E and F is determined.

Scholium. In all the foregoing problems, the angles on the ground are supposed to be taken by means of a *theodolite.* If the *sextant* be employed for that purpose, such angles, when

oblique, must be reduced by calculation to their projection on the horizontal plane *.

In surveying an extensive country, a base is first carefully measured, and the prominent distant objects are all connected with it, by a series of triangles. To avoid, in practice, the multiplication of errors, these triangles should be chosen, as nearly as possible, equilateral.—After a similar method, large estates are the most accurately planned and measured †.

The vertical angles employed in the mensuration of heights, being estimated from the varying direction of the level or the plummet, will evidently, when the stations are distant, require some correction. Let the points A and B represent two remote objects, and C their centre of gravitation; with the radius CA describe a circle, draw CB cutting the circumference in D and E, and join EA and AD. The converging lines AC and BC will indicate the direction of the plummet at A and B, the intercepted arc AD, will trace the contour of a quiescent fluid, and the tangent AZ, being applied at A, will mark the line of the horizon from that station. Wherefore the vertical angle observed at A is only ZAB, which is less than the true angle DAB, by the exterior angle DAZ. But (III. 25. El.) DAZ being equal to the angle AED in the alternate segment, is (III. 17. El.) equal to half the angle ACD at the centre. Hence the true vertical angle at any station will be found, by adding to the observed angle half the measure of the intercepted arc; and this measure depending on the curvature of the earth, which is neither uniform nor quite regular, must be deduced, for each particular place, from the length of the corresponding degree of latitude.



* See Note LXXVII. † See Note LXXVIII.

Such nicety, however, is very seldom required. It will be sufficiently accurate in practice to assume the mean quantities, and to consider the earth as a globe, whose circumference is 24,856 miles, and diameter 7,912. The arc of a minute on the meridian being, therefore, equal to 6076 feet, the correction to be added to the observed vertical angle must amount to one second, for every 69 yards contained in the intervening distance.

The quantity of depression ZD below the horizon is hence easily computed; for (III. 32. El.) $AZ^2 = EZ.ZD$, or very nearly ED.ZD; and consequently the depression of an object is proportional to the square of its distance AZ. In the space of one mile, this depression will amount to $\frac{5280}{7912}$ parts of a foot; and generally, therefore, it may be expressed in feet, by two-thirds of the square of the distance in miles.

But the effect of the earth's curvature is modified by another cause, arising from optical deception. An object is never seen by us in its true position, but in the direction of the ray of light which conveys the impression. Now the luminous particles, in traversing the atmosphere, are, by the force of superior attraction, refracted or bent continually towards the perpendicular, as they penetrate the lower and denser *strata*; and consequently they describe a curved track, of which the last portion, or its tangent, indicates the apparent elevated situation of a remote point. This trajectory, suffering almost a regular inflexure, may be considered as very nearly an arc of a circle, which has for its radius six times the radius of our globe. Hence, to correct the error occasioned by refraction, it will only be requisite to diminish the effects of the earth's curvature by one-sixth part, or to deduct, from the vertical angles, the twelfth part of the measure of the intervening terrestrial arc. The quantity of horizontal refraction, however, as it depends on the density of the air at the surface, is extremely variable, especially in our unsteady climate.

NOTES

AND

ILLUSTRATIONS.

NOTES

AND

ILLUSTRATIONS.

Note I.—Page 3.

The primary objects which Geometry contemplates are, from their nature, incapable of decomposition. No wonder that ingenuity has only wasted its efforts to define such elementary notions. It appears more philosophical to invert the usual procedure, and endeavour to trace the successive steps by which the mind arrives at the principles of the science. Though no words can paint a simple sound, this may yet be rendered intelligible, by describing the mode of its articulation.

The founders of mathematical learning among the Greeks were in general tinctured with a portion of mysticism, transmitted from Pythagoras, and cherished in the school of Plato. Geometry became thus infected at its source. By the later Platonists, who flourished in the Museum of Alexandria, it was regarded as a pure intellectual science, far sublimed above the grossness of material contact. Such visionary metaphysics could not impair the solidity of the superstructure, but did contribute to perpetuate some misconceptions, and to give a wrong turn to philosophical speculation. It is full time to restore the sobriety of reason. Geometry, like the other sciences which are not concerned about the operations of mind, must ultimately rest on external observation. But those ultimate facts are so few, so distinct, and obvious, that the subsequent train of reasoning is safely

pursued to unlimited extent, without ever appealing again to the evidence of the senses. The science of Geometry, therefore, owes its perfection to the extreme simplicity of its basis, and derives no visible advantage from the artificial mode of its construction. The axioms are rejected, as being totally useless and rather apt to produce obscurity.

The term *surface*, in Latin *superficies*, and in Greek επιφανεια, conveys a very just idea, as marking the mere expansion which a body presents to our sense of sight. *Line*, or γραμμα, signifies a *stroke*; and, in reference to the operation of writing, it expresses the *boundary* or contour of a figure. A straight line has two *radical properties*, which are distinctly marked in different languages. It holds the same undeviating course,—and it traces the shortest distance between its extreme points. The first property is expressed by the epithet *recta* in Latin, and *droite* in French; and the last seems intimated by the English term *straight*, which is evidently derived from the verb *to stretch*. Accordingly Proclus defines a straight line as *stretched* between its extremities— ἡ επ' ακρων τεταμενη.

The word *point* in every language signifies a *mark*, thus indicating its essential character, of denoting position. In Greek, the term στιγμα was first used; but, this being degraded in its application, the diminutive σημειον, formed from σημα, a *signal*, came afterwards to be preferred. The neatest and most comprehensive description of a *point* was given by Pythagoras, who defined it "a monad having position." Plato represents the *hypostasis*, or constitution of a point, as *adamantine*; finely alluding to the opinion which then prevailed, that the diamond is absolutely indivisible, the art of cutting this refractory substance being the discovery of modern ages.

The conception of an *angle* is one of the most difficult perhaps in the whole compass of Geometry. The term corresponds, in most languages, to *corner*, and therefore exhibits a most imperfect picture of the object. Apollonius defined it to be "the collection of space about a point." Euclid makes an angle to consist in "the mutual inclination, or κλισις, of its containing lines,"—a definition which is obscure and altogether defective. In strictness, this can apply only to acute angles, nor does it give any idea of angular magnitude; though

this really is as capable of augmentation as the magnitude of lines themselves. It is curious to observe the shifts to which the author of the Elements is hence obliged to have recourse. This remark is particularly exemplified in the 20th and 21st Propositions of his Third Book. Had Euclid been acquainted with Trigonometry, which was only begun to be cultivated in his time, he would certainly have taken a more enlarged view of the nature of an angle.

In the definition of *reverse angle*, I find that I have been anticipated by the famous Stevin of Bruges, who flourished about the end of the sixteenth century. It is satisfactory to have the countenance of such respectable authority.

Note II.—Page 9.

A *square* is commonly described as having *all* its angles right. This definition errs however by excess, for it contains more than what is necessary. The original Greek, and even the Latin version, by employing the general terms *ὀρθογώνιον*, and *rectanglum*, dexterously, avoided that objection. The word *rhombus* comes from *ῥυμβεῖν*, *to sling*, as the figure represents only a quadrangular frame disjointed.

It scarcely deserves notice, but I will anticipate the objection which may be brought against me, for having changed the definition of *trapezium*. The fact is, that I have only restricted the word to its appropriate meaning, from which Euclid had, according to Proclus, taken the liberty to depart. In the original, it signifies *a table*; and hence we learn the prevailing form of the tables used among the Greeks. Indeed the ancients would appear to have had some predilection for the figure of the trapezium, since the doors now seen in the ruins of the temples at Athens are not exactly oblong, but wider below than above.

Language is capable of more precision, in proportion as it becomes copious. As I have confined the epithet *right* to angles, and *straight* to lines, I have likewise appropriated the word *diagonal* to rectilineal figures, and *diameter* to the circle. In like manner, I have restricted the term *arc* to a portion of the circumference, its synonym *arch* being assigned to architecture. For the same reason, I have adopted

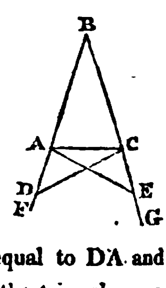
the term *equivalent*, from the celebrated Legendre, whose *Elemens de Geometrie* is one of the ablest works that has appeared in our times. These distinctions evidently tend to promote perspicuity, which is the great object of an elementary treatise.—Euclid and all his successors define an isosceles triangle to have *only* two equal sides, which would absolutely exclude the equilateral triangle. Yet the equilateral triangle is afterwards assumed by them to be a species of isosceles triangle, since the equality of its angles is at once inferred as a corollary from that of the angles at the base of an isosceles triangle. This inadvertency, slight as it may appear, is now avoided.

Note III.—Page 18.

This proposition may be very simply demonstrated, in the same manner as its converse, by a direct appeal to superposition or mental experiment. For suppose a copy of the triangle ABC were inverted and applied to it, the side BA being laid on BC, the side BC again will evidently lie on BA, and the base AC coincide with CA. Consequently the angle BAC, occupying now the place of BCA, must be equal to this angle.

It may be worth while to remark, that Euclid's demonstration, which, being placed near the commencement of the Elements, has from its intricacy been styled the *Pons Asinorum*, is in fact essentially the same. This will readily appear on a review of the several steps of the reasoning:—

The sides BA and BC of the isosceles triangle being produced, the equal segments AD and CE are assumed, and AE, CD joined. 1. The complex triangles ABE and CBD are compared: The sides AB and BC are equal, and likewise BE and BD, which consist of equal parts, and the contained angles EBA and DBC are the same with DBE; whence (I. 3.) these triangles are equivalent, and the base AE equal to CD, the angle BAE equal to BCD, and the angle BEA to BDC. 2. The additive triangles CAE and ACD are next compared: The sides EC and EA being equal to DA and DC, and the contained angle CEA equal to ADC, the triangles are



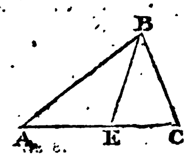
(I. 3.) equivalent, and therefore the angle CAE is equal to ACD. 3. Lastly, since the whole angle BAE is equal to BCD, and the part CAE to ACD, the remainder BAC must be equal to BCA.

Now this process of reasoning is at best involved and circuitous. The compound triangles ABE and CBD consist of the isosceles triangle ABC joined to each of the appended triangles ACE and CAD; when therefore, as the demonstration implies, ABE is laid on CBD, the common part ABC is reversed, or it is applied to CBA; and the other part ACE is laid on CAD. But the superposition of ABC or CBA is easily perceived by itself; nor is the conception of that inverted application anywise aided by having recourse to the superposition, first of the enlarged triangles ABE and CBD, and then contracting these by the superposition of the subsidiary triangles ACE and CAD. In this, as in some other instances, Euclid has deceived himself, in attempting a greater than usual strictness of reasoning.

Note IV.—Page 20.

This proposition may be demonstrated otherwise.

Draw (I. 5. El.) BE bisecting the angle ABC. The angle BEA (I. 8. El.) is greater than the interior angle EBC or EBA, and therefore (I. 14. El.) the side AB is greater than AE. In like manner, the angle BEC is greater than the interior angle EBA or EBC, and consequently (I. 14. El.) the side CB is greater than CE. Wherefore the two sides AB and CB, being each of them greater than the adjacent segments AE and CE, are together greater than the whole base AC.



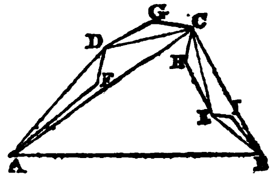
Note V.—Page 20.

From this property of the sides of a triangle, may be derived the generic character of a straight line:

The shortest line that can be drawn between two points, is a straight line.

Let the points A and B be connected by straight lines joining an intermediate point C; and the two sides AC and BC of the triangle ACB are greater than AB (I. 15.). Now let a third point D be in-

terposed between A and C; and because AD and DC are together greater than AC, add BC to both, and the three lines AD, DC, and CB are greater than AC and BC, and consequently still greater than AB. Again, suppose a fourth point E to connect B with C; and the sides BE and CE of the triangle BCE being greater than BC, the four straight lines AD, DC, CE, and EB are together, by a still farther access, greater than AB. By thus repeatedly multiplying the interjacent points, two sides of a triangle will at each successive step come in place of a third side, and consequently the aggregate polygonal or crooked line AFDGCHEIB will acquire continually some farther extension. Nay, since there is no limit to the possible number of those connecting points, they may approach each other nearer than any assignable interval; and consequently the proposition is also true in that extreme case where the boundary is a curve line, or of which no portion can be deemed rectilineal.

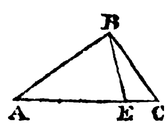


The proposition now demonstrated is commonly assumed as an axiom. It is indeed forced upon our earliest observation, being suggested by the stretching of a cord, and other familiar occurrences in life. But thus to multiply principles, appears quite unphilosophical. The two radical properties of a straight line—the congruity of its parts—and its shortness of trace—are distinct, though connected. The latter is shown to be the necessary consequence of the former; but it would be impossible, by any direct process, to infer the uniformity of straight lines, from their marking out the nearest routes.

Note VI.—Page 20.

This proposition may be otherwise demonstrated.

Join BE. The angle BEC (I. 8. El.) is greater than ABE or (I. 11. El.) AEB, which again (I. 8. El.) is greater than CBE; wherefore (I. 14. El.) the side BC is greater than CE, or the difference between AB and AC.



In the demonstration, I could not avoid introducing the consideration of limits. This will occasion, I presume, no material difficulty, since the reasoning is actually the same as that by which our most familiar conceptions are gradually expanded.

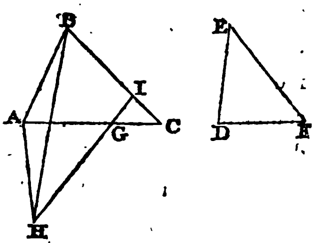
Note VII.—Page 23.

The ingenious Mr Thomas Simpson has very justly remarked, in his Elements of Geometry, that the demonstration which Euclid gives of this proposition is defective, since it assumes that the point G must lie below the base AC. He has therefore legitimately supplied the deficiency of the proof; and it is surprising that so rigorous a geometer as Dr Robert Simson, should have so far yielded to his prejudices, as to resist such a decided improvement. The demonstration inserted in the text appears to be rather simpler and more natural than that of Mr T. Simpson.

Note VIII.—Page 23.

This proposition is capable of being demonstrated directly.

Let the triangles ABC and DEF have the sides AB and BC equal to DE and EF, but the base AC greater than DF; the vertical angle ABC is greater than DEF.



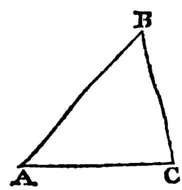
From AC cut off AG equal to DF, construct (I. 1.) the triangle AHG having the sides AH and GH equal to AB and BC or DE and EF, join HB, and produce HG to meet BC in I.

Because HI is greater than HG, it is greater than the equal side BC, and therefore much greater than BI. Consequently the opposite angle IBH of the triangle BIH is (I. 13.) greater than BHI. But AB being equal to AH, the angle HBA is (I. 11.) equal to BHA, and therefore the two angles IBH and HBA are greater than IHB and BHA, that is, the whole angle CBA is greater than IHA or GHA. And since the sides of the triangle AGH are by construction equal to those of EDF, the corresponding angle AHG is equal to DEF (I. 2.); and hence the angle ABC, which is greater than AHG, is likewise greater than DEF.—In like

manner this may be demonstrated, if BH should fall without the figure.

Note IX.—Page 26.

It is not difficult to perceive that the whole structure of *geometry* is grounded on the simple comparison of triangles. The conditions which fix the equality of those elementary figures, are all contained in the 2d, 3d, 21st and 22d propositions of the first Book. These fundamental theorems derive their evidence from the mere *superposition* of the triangles themselves, which, in reality, is nothing but an *ultimate* appeal, though of the easiest and most familiar kind, *to external observation*. The same conclusions however might be deduced more concisely, from the circumstances which must determine the constitution of an individual triangle. Suppose AB, BC, and AC, any one of which is shorter than the other two conjoined, to be three inflexible rods moveable at pleasure. (1.) Place them with their ends meeting each other, and they will evidently rest in the same position, and contain a distinct triangle,—which corresponds to Prop. 2. (2.) Having joined the rods AB and BC at B, continue to open them at that point, till they form a given vertical angle ABC; their position then becomes fixed, and consequently determines the rod AC which connects their extremities and completes the triangle. This inference evidently agrees with Prop. 3. (3.) While the rod AC retains its place, let two rods AB and CB of unlimited length, and applied at the ends A and C, be opened gradually till the one forms with AC a given angle CAB, and the other a given angle ACB; it is evident that AB and BC will then rest crossing each other in those positions, and containing a determinate triangle, of which the vertex B is their point of mutual intersection. This property corresponds with Prop. 21. (4.) Let the rod AB of a given length make a given angle with the unlimited rod AC, and applying at the end B another given rod, turn this gradually round till it meets AC. If BC exceeds the distance of B from AC, it will evidently, after stretching beyond AC, again come to meet that boundary. With such conditions, therefore, the rods might contain two



determinate triangles, the one acute and the other obtuse, and which are hence distinguished from each other, by the additional character of *affection*. This qualified property, omitted in most elementary works, is yet of extensive application, and was requisite to complete the conditions of the equality of triangles. It corresponds with Prop. 23.

The four preceding theorems are reducible, however, to a single property, which includes all the different requisites to the equality of triangles. The sides of a triangle are obviously independent of each other, being subject to this condition only, that any one of them shall be less than the remaining two sides. But since all the angles of a triangle are together equal to two right angles, the third angle must, in every case, be the necessary result of the other two angles. A triangle has, therefore, only five original and variable parts—the three sides and two of its angles. *Any three of these parts being ascertained, the triangle is absolutely determined.* Thus—when (1.) all the three sides are given,—when (2.) two sides and their contained angle are given,—when (4.) two sides and an opposite angle are given, with the affection of the triangle, or when (3.) one side and two angles, and thence the third angle are given,—the triangle is completely marked out.

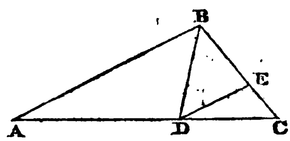
M. Legendre, in a very elaborate note to his *Elemens de Geometrie*, has sought, with much ingenuity, to deduce *à priori* the radical properties of triangles, from the theory of *functions*. But, like other similar attempts, his investigation actually involves in it a latent assumption. This subtle logician sets out with the principle which would seem almost intuitive, that a triangle is determined when the base and its adjacent angles are given. The vertical angle, therefore, depends wholly on these data—the base and its adjacent angles. Call the base c, its adjacent angles A, B, and the vertical angle C. This third angle being derived from the quantities A, B and c, must be a determinate function of them, or formed from their combination. Whence, adopting his notation, $C = \varphi : (A, B, c)$. But the line c is of a nature heterogeneous to the angles A and B, and therefore cannot be compounded with these quantities. Consequently $C = \varphi : (A, B)$, or the vertical angle is simply a function of the angles A and B at the base, and hence the third angle of a triangle must depend wholly on the other two.

To a speculative mathematician this argument is very seductive, though it will not bear a rigid examination. Many quantities

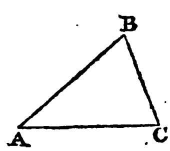
in fact appear to result from the combined relation of other quantities that are altogether heterogeneous. Thus, the space which a moving body describes, depends on the joint elements of time and velocity, things entirely distinct in their nature; and thus, the length of an arc of a circle is compounded of the radius, and of the angle it subtends at the centre, which are obviously heterogeneous magnitudes. For aught we previously knew to the contrary, the base c might, by its combination with the angles A and B, modify their relation, and thence affect the value of the vertical angle C. In another parallel case, the force of this remark is easily perceived. Thus, when the sides a, b and their contained angle C are given, the triangle is determined, as the *simplest* observation shows. Wherefore the base c is derived solely from these data, or $c=\phi:(a, b, C)$. But the angle C, being heterogeneous to the sides a and b, cannot coalesce with them into an equation, and consequently the base c is simply a function of a and b, or it is the necessary result merely of the other two sides. Such is the extreme absurdity to which this sort of reasoning would lead! In both of these instances, indeed, the conclusion is admitted by implication, only in the one it is consistent with truth, while in the other it is palpably false.—That such an acute philosopher could overlook the fallacy of his argument, can only be ascribed to the influence which peculiar trains of thought acquire over the mind, and to the extreme facility with which elementary principles insinuate and blend themselves with almost every process of reasoning.

Admitting, however, what the slightest inspection readily confirms, that the third angle is merely derived from the other two, M. Legendre demonstrates with great elegance, the property that the three angles of a triangle are equal to two right angles. Letting fall from the right angle a perpendicular on the hypotenuse, he divides any right-angled triangle into two subordinate triangles, which have each of them two angles equal to those of the original triangle; whence the acute angles of that triangle are alternately equal to the angles which compose the right angle. But every triangle may be divided into two right-angled triangles, by letting fall a perpendicular from the vertex on the base, and consequently the acute angles of both these triangles, and which form the angles at the base, and the vertical angle of the primary triangle,—are together equal to two right angles.

This theorem may be proved somewhat more directly. In the triangle ABC, let the angle CBA be greater than ACB, and draw BD, and then DE, making the angles ABD and BDE each equal to ACB. The triangles ABC and ADB having the common angle BAC and the angle ACB equal to ABD, their third angles ABC and ADB must be equal. But the triangles BCD and BDE have also a common angle CBD, and equal angles DCB and BDE; whence the third angle BDC is equal to BED, and therefore the supplementary angle ADB, equal to ABC. is equal to DEC. Again, the triangles ABC and DEC having two common or equal angles, their third angles BAC and EDC are equal; wherefore the three angles ABC, BCA and BAC of the original triangle, are respectively equal to BDA, BDE and EDC, and hence equal to two right angles.—If the triangle ABC be equiangular, divide it into two scalene triangles ABD and CBD, the angles of which, or the angles of the original triangle, together with the adjoining angles ADB and BDC, must be equal to four right angles, and consequently the angles of that triangle are equal to two right angles.



But the proposition is easily derived from another view of the subject. If we suppose a ruler turning about the point A, to change its direction AC into AB, then opening at B till it gains the direction BC, and finally wearing about the point C till it acquires the opposite position CA; thus changing its direction with respect to a remote object, by three successive openings all to the same side, the ruler, being now reversed, must have performed half a circuit; that is, the three angles of a triangle, which constitute those openings, are equal to two right angles.



The profound geometer already quoted, pursuing his refined argument, has, from the consideration of homogeneous quantities, likewise attempted to deduce the proportionality of the sides of equiangular triangles. But in this abstruse research, assumptions are still disguised and mixed up with the process of induction. Such indeed must be the case with every kind of reasoning on mathematical or physical objects, which proceeds *à priori*, without appealing,

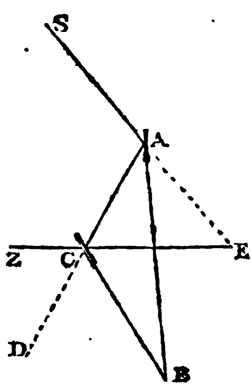
at least in the first instance, to external observation. Of this kind, are some of those ingenious analytical investigations respecting the laws of motion and the composition of forces. The principle of *sufficient reason*, introduced by Leibnitz, appears to be nothing but an artificial mode of dressing out an assumption, and which the celebrated Boscovich has well exposed in his excellent *notes to* a didactic poem by Stay, entitled *Philosophia Recentior.*

Note X.—Page 27.

The subject of parallel lines has exercised the ingenuity of modern geometers; for Euclid had only endeavoured to evade the difficulty, by styling the fundamental proposition an axiom. The investigation now given seems to be one of the best adapted to the natural progress of discovery. It is almost ridiculous to scruple about admitting the idea of motion, which I have employed for the sake of clearness. But even that futile objection might be obviated, by considering merely the successive positions of the straight line extending through the given point.

Note XI.—Page 32.

That invaluable instrument, Hadley's quadrant, is founded on the second corollary, annexed as an obvious consequence of the proposition. A ray of light SA, from the sun, impinging against the mirror at A, is reflected at an angle equal to its incidence; and now striking the half-silvered glass at C, it is again reflected to E, where the eye likewise receives, through the transparent part of that glass, a direct ray from the boundary of the horizon. Hence the triangle AEC has its exterior angle ECD and one of its interior angles CAE, respectively double of the exterior angle BCD and the interior angle CAB, of the triangle ABC; wherefore the remaining interior angle AEC, or SEZ, is double of ABC; that is, the altitude of the sun above the horizon is double of the inclination of the two mirrors. But the glass at C remaining fixed, the mirror at A is attached to a moveable index, which marks their inclination.

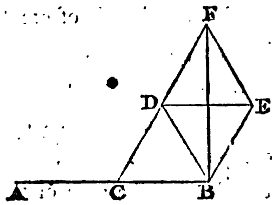


The same instrument, in its most improved state, and fitted with a telescope, forms the sextant, which, being admirably calculated for measuring angles in general, has rendered the most important services to geography and navigation.

Note XII.—Page 35.

This problem is generally constructed somewhat differently.

In AB take any point C, and on BC (I. 1. cor.) describe an equilateral triangle CDB, on its side DB, another DEB; and on DE the side of this, a third equilateral triangle DFE; join the last vertex F with the point B; and BF is the perpendicular required.



Because the triangles CDB and DBE are equilateral, the angles CBD and DBE are each of them equal to two third parts of a right angle (I. 32. cor.); and the triangles BDF, BEF, having the sides BD, DF equal to BE, EF, and the side BF common, are (I. 2.) equal, and consequently the angles FBD and FBE are equal, and each of them the half of DBE. The angle FBD, being therefore one-third part of a right angle, and the angle DBA two-third parts, the whole angle FBC must be an entire right angle, or the straight line BF is perpendicular to AB.

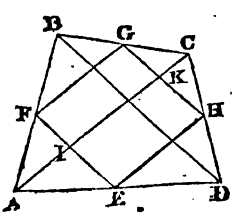
Note XIII.—Page 42.

From this proposition the following theorem is easily derived:

Straight lines joining the successive middle points of the sides of a quadrilateral figure, form a rhomboid.

If the sides of the quadrilateral figure ABCD be bisected, and the points of section joined in their order; EFGH is a rhomboid.

For draw AC, BD. And because FG bisects AB, BC, it is (II. 4. El.) parallel to AC; and for the same reason, EH, as it bisects AD and DC, is parallel to AC. Wherefore FG is parallel to EH (I. 30.). In like manner, it is proved that EF is parallel to HG; and consequently the figure EFGH is a rhomboid or parallelogram.



It is likewise evident, that the inscribed rhomboid is half of the quadrilateral figure; for IG is half of the triangle ABC (II. 4. cor.), and IH is half of the triangle ADC.

Note XIV.—Page 43.

This problem is of great use in practical geometry. The plan, for instance, of any grounds however irregular, is divided into a number of triangles, which are successively reduced to a simple triangle, and this again is converted (by II. 7.) into a rectangle. Instead of computing, therefore, each component triangle, it may be sufficient to calculate the area of the final triangle or rectangle.

Note XV.—Page 46.

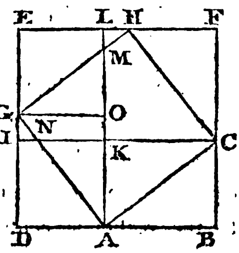
On this proposition is founded the method of *offsets*, which enters so largely into the practice of land-surveying. In measuring a field of a very irregular shape, the principal points only are connected by straight lines forming sides of the component triangles, and the distance of each remarkable flexure of the extreme boundary is taken from these rectilineal traces. The exterior border of the polygon is therefore considered as a collection of trapezoids, which are measured by multiplying the mean of each pair of offsets or perpendiculars into their base or intermediate distance.

Note XVI.—Page 48.

This famous proposition appears to have been brought from the East by Pythagoras. The method here given of demonstrating it, from the transposition of the several parts of the figure, is ascribed to the Persian astronomer Nassir Eddin, who flourished in the thirteenth century of our æra, under the munificent patronage of the conqueror Zingis Khan.

It may gratify the young student in Geometry to see the mode of

performing this dissection. Having drawn GO parallel to IK, place the triangle CKA on CFH, invert the triangle GOA or ADG, place the triangle GOM on AKN, and transfer the small triangle GIN to HLM. In this way, the square AGHC is transformed into the two squares CKLF and ADIK. By reversing the process, the squares of the sides of the right-angled triangle may be compounded into the single square of the hypotenuse.



Note XVII.—Page 49.

It was a favourite speculation with the Greek geometers, to express numerically the sides of a right-angled triangle. The rules which they delivered for that purpose are equally simple and ingenious. For the sake of conciseness, it will be convenient, however, to adopt the language of symbols. Let n denote any odd number; then,

according to Pythagoras, $n, \frac{n^2-1}{2}$ and $\frac{n^2+1}{2}$, or

according to Plato, $2n, n^2-1$ and n^2+1, will represent the perpendicular, the base, and hypotenuse, of a right-angled triangle. Thus, n being supposed equal to 3, the numbers thence resulting are 3, 4, and 5, or 6, 8, and 10. These expressions are fundamentally the same, and are easily derived from Prop. 19. Book II.: For $(n^2+1)^2-(n^2-1)^2=((n^2+1)+(n^2-1))((n^2+1)-(n^2-1))=2n^2\times2=(2n)^2$.

Note XVIII.—Page 51.

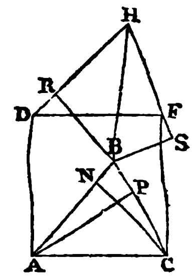
An elegant proposition derived from this, deserves a place in an elementary work:

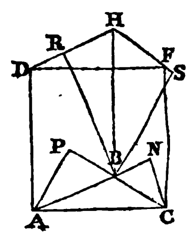
In any triangle, the square described on the base, is equivalent to the rectangles contained by the two sides and their segments intercepted from the base by perpendiculars let fall upon them from its opposite extremities.

Let the perpendiculars AP, CN be let fall from the points A, C upon the opposite sides BC and AB of the triangle ABC; the square of AC is equivalent to the rectangles contained by AB, AN and by BC, CP.

For complete the rhomboids ADHB and CFHB, and let fall the perpendiculars BR and BS upon DH and FH.

It is manifest, (II. 15. El.) that the rhomboids AH and CH are equivalent to the square of AC. But the rhomboid AH is equivalent to the rectangle contained by AB and BR (II. 1. cor.). Comparing the triangles BHR and ACN; the angle BRH, being a right angle, is equal to ANC; and the two acute angles BHR and RBH, being together equal to a right angle, are equal to DAN and NAC; but DAB is equal to DHB (I. 27.), whence the angle RBH is equal to NAC. These triangles BHR and ACN, having thus two angles respectively equal, and the corresponding side BH in the one equal to AD or AC in the other, are therefore equal (I. 21.), and consequently the side BR is equal to AN. The rectangle AB and BR, which is equivalent to the rhomboid AH, is hence equivalent to the rectangle contained by AB and AN (II. 1. cor.).





In the same manner, it may be demonstrated, by comparing the triangles BHS and PAC, that the rectangle under BC and BS which is equivalent to the rhomboid CH, is equivalent to the rectangle contained by BC and CP. Wherefore the two rectangles of AB, AN and BC, CP are together equivalent to the square described on AC.

If the triangle ABC be right-angled at the vertex B, the perpendiculars CN and AP will evidently meet at the vertex, and consequently the rectangles AB, AN and BC, CP will become the squares of AB and BC. And hence the beautiful Proposition II. 11. is derived, being only a remarkable case of a much more general property.

Note XIX.—Page 51.

Since rectangles correspond to numerical products, the properties of the sections of lines are easily derived from symbolical arithmetic:

1. In Prop. 16. let AC be denoted by a, and the segments of AB by b, c and d; then $a(b+c+d)=ab+ac+ad$.

2. In Prop. 17. let the two lines be denoted by a and b; then $(a+b)^2=a^2+b^2+2ab$.

3. In Prop. 18. let the two lines be denoted by a and b; then $(a-b)^2=a^2+b^2-2ab$.

4. In Prop. 19. let the two lines be denoted by a and b; then $(a+b)\,(a-b)=a^2-b^2$.

5. In Prop. 20. let the segments of the compound line be denoted by a, b and c; then $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$.

6. In Prop. 21. let the two lines be denoted by a and b; then

$$a^2+b^2=\tfrac{1}{2}(a+b)^2+\tfrac{1}{2}(a-b)^2=2\left(\frac{a+b}{2}\right)^2+2\left(\frac{a-b}{2}\right)^2.$$

7. In Prop. 22. let the whole line be denominated by a, and its greater segment by x; then $x^2=a(a-x)$, and $x^2+ax=a^2$, whence $x=\pm\sqrt{\frac{5a^2}{4}}-\frac{a}{2}=\pm a(\sqrt{\tfrac{5}{4}}-\tfrac{1}{2})$. Hence, if unit represent the whole line, the greater segment is .61803398428, &c. and the smaller segment .38196601572, &c.

From Cor. 1. an extremely neat approximation is likewise obtained. Assuming the segments of the divided line as at first equal and denoted each by 1, these successive numbers will result from their continued summation:

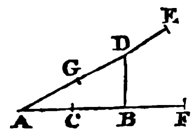
1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c.

If the original line, therefore, contained 144 equal parts, its greater segment would include 89, and its smaller segment 55 of these parts very nearly.

Note XX.—Page 59.

This problem may, however, be constructed somewhat differently, without employing the collateral properties.

For bisect AB in C (I. 7.), draw (I. 5. cor.) the perpendicular BD equal to BC, join AD and continue it until DE be equal to DB or BC, and on AB produced take AF equal to AE: The line AF is the required extension of AB. For make DG equal to DB or BC; and because (II. 19. cor. 2.) the rectangle EA, AG together with the square of DG or DB, is equivalent to the square of DA, or to the squares of AB and DB; the rectangle EA, AG, or FA, FB is equivalent to the square of AB.



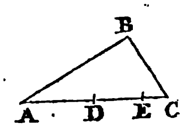
Note XXI.—Page 60.

A neat proposition may be subjoined.

If, from the hypotenuse of a right-angled triangle, portions be cut off equal to the adjacent sides; the square of the middle segment thus formed, is equivalent to twice the rectangle contained by the extreme segments.

Let ABC be a triangle which is right-angled at B; from the hypotenuse AC, cut off AE equal to AB, and CD equal to CB: Twice the rectangle under AD and CE is equivalent to the square of DE.

For the straight line AC being divided into three portions, the squares of AE and CD, together with twice the rectangle AD, CE are equivalent to the squares of AC and DE (II. 20. cor.). But the squares of AB and BC, or those of AE and CD, are equivalent to the square of AC (II. 11.). There consequently remains twice the rectangle AD, CE equivalent to the square of DE.



By an inverse process of reasoning it will appear, that if twice the rectangle AD, CE be equal to the square of DE, the straight line AC, so composed, is the hypotenuse of a right-angled triangle, of which AB and BC are the sides.

This proposition will furnish another convenient method of discovering the numbers which represent the sides of a right angled triangle: For since $DE^2 = 2AD \times CE$, it is evident that $\frac{1}{2}DE^2 = AD \times CE$; and consequently, expressing DE by a whole number, and resolving $\frac{1}{2}DE^2$ into the factors AD and CE, $AD+DE$ and $CE+DE$ will represent the two sides, and $AD+CE+DE$ the hypotenuse. Thus, if 2 be taken, the factors of half its square are 1 and 2, which produce the numbers 3, 4, and 5. Again, if 4 be assumed, the factors are 2 and 4, or 1 and 8; whence result these numbers, 6, 8 and 10, or 5, 12 and 13. In this way, a very great variety of numbers can be found, to express the sides of a right-angled triangle.

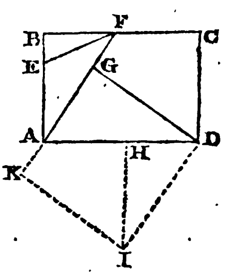
Note XXII.—Page 61.

This proposition is of great use in practical geometry, since it enables us to divide a triangle, of which all the sides are given into two right-angled triangles, by determining the position and consequently the length of the perpendicular.

Note XXIII.—Page 63.

From this corollary is derived a very simple construction of the problem, "to find a square equivalent to a given rectangle."

Let ABCD be the given rectangle, of which the side AD is greater than AB. In AB or its production, take AE equal to the half of AD and place it from E to F; then AF being joined, is the side of the equivalent square. For (II. 26. cor. El.) since the sides AE and EF of the triangle AEF are equal, the square of AF is equivalent to the rectangle under twice AE and AB, that is, from the construction, the rectangle under AD and AB.



The same construction might likewise be deduced from the second demonstration of the celebrated property of the right-angled triangle. For, in the figure of page 48, suppose BO were drawn to the hypotenuse AC, making an angle ABO equal to BAO or BAC; since the two acute angles are together equal to a right angle, the angle BCA

is equal to the remaining portion CBO of the right angle at B, and consequently the triangles AOB and COB are isosceles, and the sides OA, OB and OC all equal. Wherefore AB, the side of a square equivalent to the rectangle ADMN or that under AK and AN, is determined by making AO equal to the half AK or AC and inserting it from O to B.—The inspection of the same figure also points out the mode of dissecting the rectangle, and thence compounding the square; for a perpendicular let fall from K on AB is evidently equal to GB or AB. Hence, on AF, in the original construction, let fall the perpendicular DG, transpose the triangle FBA in the situation DHI, and slide the quadrilateral portion into the place of KAHI; the rectangle ABCD is now transformed into the square KGDI.—A slight modification will be required when AB is less than the half of AD.

In this construction of the problem, the application of the circle which (III. 33. El.) is indispensably required, is only not brought into view.—When the side AD is double of AB, the point G coincides with F, and the rectangle is resolved into three triangles, which combine to form a square.

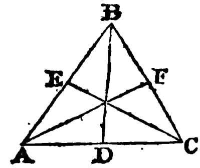
Note XXIV.—Page 64.

The following theorem is demonstrated from the same principles:

If straight lines be drawn from the angular points of a triangle to bisect the opposite sides, thrice the squares of these sides are together equivalent to four times the squares of the bisecting lines.

Let the sides of the triangle ABC be bisected in D, E, and F, and straight lines drawn from these points to the opposite vertices; thrice the squares of the sides AB, BC, and AC are together equivalent to four times the squares of BD, CE and AF.

For, by Prop. II. 25. the squares of AB, BC are equivalent to twice the square of BD and twice the square of AD, that is, half the square of AC; the squares of BC, AC are equivalent to twice the squares of CE and half the square of AB; and the squares of AC, AB are equivalent to twice the square of AF and half the square of BC.



Whence the squares of the sides of the triangle, repeated twice, are

equivalent to twice the squares of BD, CE, and AF, with half the squares of the sides of the triangle. Consequently four times the squares of AB, BC, and AC are equivalent to four times the squares of BD, CE, and AF, with once the squares of AB, BC, and AC; wherefore thrice the squares of the sides AB, BC, and AC are together equivalent to four times the squares of the bisecting lines BD, CE, and AF.

Note XXV.—Page 65.

This general theorem seems to have been first given by the illustrious Euler in the Petersburg Memoirs. It evidently comprehends the proposition which stands immediately before it; for when the quadrilateral figure becomes a rhomboid, the diagonals bisect each other, and the middle points E and F coincide; whence the squares of all the sides are equivalent simply to the squares of those diagonals.—If this rhomboid again becomes a rectangle, it will have equal diagonals, and consequently, as in the 11th Proposition of the second book, the squares of the sides of a right angled triangle are equivalent to the square of the hypotenuse.

Note XXVI.—Page 81.

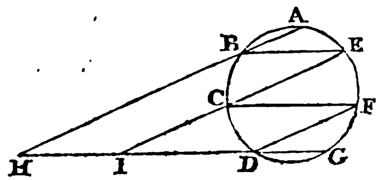
Hence angles are sometimes measured by a circular instrument, from a point in the circumference, as well as from the centre.—On the next proposition depends the construction of amphitheatres; for the visual magnitude of an object is measured by the angle which it subtends at the eye, and consequently the whole extent of the stage will be seen with equal advantage by every spectator seated in the same arc of a circle.

Note XXVII.—Page 83.

If, on each side of any point in the circumference of a circle, equal arcs be repeated; the chords which join the opposite points of section will be together equal to the last chord extended till it meets a straight line drawn through the middle point and either extremity of the first chord.

Let DAG be the circumference of a circle, in which the arcs AB, BC, CD on the one side of a point A, and the corresponding arcs AE, EF, FG on the other side, are all assumed equal; the chords BE, CF, and DG, are together equal to the line GH, formed by extending GD till it meets the production of AB.

For join FD and CE, and produce this to meet GH in the point I.



Because the arcs BC and CD are equal to EF and FG, the chords BE, CF, and DG are parallel; but, for the same reason, since the arcs BC and CD are equal to AE and EF, the chords BA, CE and DF are likewise parallel. Hence the figures HBEI and ICFD are rhomboids, and therefore the extended chord GH, being composed of the segments HI, ID, and DG, is equal to the sum of their opposite chords BE, CF and DG. *It is* obvious that the same train of reasoning may be pursued to any number of equal arcs.

Note XXVIII.—Page 84.

This proposition is of some utility in practice, for an angle may be hence measured by help of a circular protractor, without the trouble of applying the centre to its vertex, or the point of concourse of the sides.—The same principle is likewise applicable to the construction of some optical instruments, calculated to measure lateral angles by the intersection of micrometer wires.

Note XXIX.—Page 85.

To *erect* a perpendicular, any point D is taken, as in Prop. 36. Book I., and from it a circle is described passing through C and B; the diameter CDF determines the position of the perpendicular BF. To *let fall* a perpendicular, draw to AB any straight line FC, which bisect in D, and from this point as a centre describe a circle through C, B and F, FB is the perpendicular required.

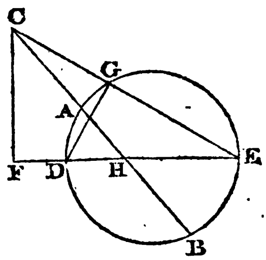
Note XXX.—Page 96.

The rectangle under the segments of a chord is greater or less than the rectangle under the segments into which a perpendicular from the point of section divides a diameter, by the square of that perpendicular —according as it lies without or within the circle.

Let the perpendicular CF be let fall from a point C in the chord ACB upon a diameter DE; the rectangle BC, CA, is greater or less than the rectangle EF, FD, by the square of the perpendicular CF, according as this lies without or within the circle.

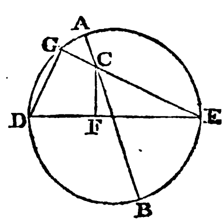
First, let the perpendicular CF lie without the circle, and join CE and DG.

The square of the hypotenuse CE is equivalent to the squares of FE and CF (II. 11.). But the square of CE is composed of the rectangles CE, EG, and CE, CG (II. 16.); and the square of FE is composed of the rectangles FE, ED, and FE, FD: Wherefore the rectangles CE, EG and CE, CG are equivalent to the rectangles FE, ED and FE, FD, together with the square of CF. And since EGD, standing in a semicircle, is a right angle (III. 22.), its adjacent angle CGD is also right, and the angle opposite to this at F is right; consequently (III. 19. cor.) a circle might be described through the four points C, G, D, F. Whence (III. 32.) the rectangle CE, EG is equivalent to FE, ED; and taking these from the terms of the former equality, there remains the rectangle CE, CG, that is, (III. 32.) AC, CB, equivalent to the rectangle FE, FD, together with the square of CF.



Next, let the perpendicular CF lie within the circle.

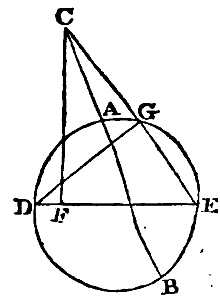
The same construction being made, the rectangle CE, EG is still equivalent to the rectangle FE, ED. But the rectangle CE, EG is (II. 16.) equivalent to the rectangle CE, CG, and the square of CE, or the squares of FE and CF; and the rectangle FE, ED is equivalent to the rectangle FE, FD and the square



of FE. From these equal quantities, therefore, take away the common square of FE, and there remains the rectangle CE, CG, or AC, CB, with the square of CF, equivalent to the rectangle FE, FD.

Lastly, if the perpendicular CF lie partly without and partly within the circle, the Proposition must be slightly modified.

The former construction being retained: Because the square of CE is equivalent to the squares of CF and FE, the rectangles CE, EG and CE, CG are together equivalent to the square of CF and the difference between the rectangle FE, ED and FE, FD; but the rectangle CE, EG is equivalent to the rectangle FE, ED, and consequently the rectangle CE, CG, or the rectangle AC, CB, is equivalent to the difference between the square of CF and the rectangle FE, FD.



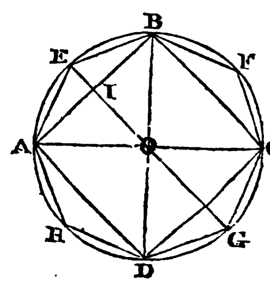
In the first case, if the square of FH be equivalent to the rectangle FD, FE, the square of CH will be likewise equivalent to the rectangle CG, CE; for the rectangle AC, CB, being equivalent to the rectangle FD, FE, or the square of FH, together with the square of CF, must (II. 11. El.) be equivalent to the square of CH.

Note XXXI.—Page 118.

The square of the side of a regular octagon inscribed in a circle, is equivalent to the rectangle contained by the radius and the difference between the diameter and the side of the inscribed square.

Let ABCD be a square inscribed in a circle, and AEBFCGDH an octagon, which is formed evidently by the bisection of the quadrants AB, BC, CD, and DA: The square of AE is equivalent to the rectangle under AO and the difference between AB and AC.

For draw the diameter EG. It is manifest, that the triangles AIO and BIO are right-angled and isosceles; and because AO is equal to EO, and AI perpendicular to it,—the square of AE (II. 26. cor. El.) is equivalent to twice the rectangle under EO and EI, or the rectangle under AO and twice EI. But EI is the difference of EO and IO,



and twice EI is, therefore, equal to the difference of twice EO or AC and twice IO or AB. Whence the square of AE, the side of the octagon, is equivalent to the rectangle under the radius and the difference of the diameter and AB the side of the inscribed square.

Note XXXII.—Page 119.

Such were the only regular polygons known to the Greeks. The inscription of all the rest has for ages been supposed absolutely to transcend the powers of elementary geometry. But a curious and most unexpected discovery was lately made by Mr Gauss, who has demonstrated, in a work entitled *Disquisitiones Arithmeticæ*, and published at Brunswick in 1801, that certain very complex polygons can yet be described merely by help of circles. Thus, a regular polygon containing 17, 257, 65537, &c. sides, is capable of being inscribed, by the application of elementary geometry; and in general, when the number of sides may be denoted by 2^n+1, and is at the same time a prime number. The investigation of this principle is rather intricate, being founded on the arithmetic of sines and the theory of equations; and the constructions to which it would lead are hence, in every case, unavoidably and most excessively complicated. Thus the cosine of the several arcs arising from the division of the circumference of a circle into seventeen equal parts, are all contained in this very involved expression:

$$-\tfrac{1}{16}+\tfrac{1}{16}\sqrt{17}+\tfrac{1}{16}\sqrt{(34-2\sqrt{17})}-$$
$$\tfrac{1}{8}\sqrt{\left(17+3\sqrt{17}-\sqrt{(34-2\sqrt{17})}-2\sqrt{(34+2\sqrt{17})}\right)}.$$

As the radicals may be taken either positive or negative, their various combinations, rightly disposed, will produce eight distinct results. Let π denote the semicircumference; then $\cos\frac{2\pi}{17}=\cos\frac{32\pi}{17}=.9324722294$, $\cos\frac{4\pi}{17}=\cos\frac{30\pi}{17}=.7390089172$, $\cos\frac{6\pi}{17}=\cos\frac{28\pi}{17}=.4457383558$, $\cos\frac{8\pi}{17}=\cos\frac{26\pi}{17}=.0922683595$, $\cos\frac{10\pi}{17}=\cos\frac{24\pi}{17}=$ $-.27366229901$, $\cos\frac{12\pi}{17}=\cos\frac{22\pi}{17}=-.6026346364$, $\cos\frac{14\pi}{17}=$ $\cos\frac{20\pi}{17}=-.8502171357$, and $\cos\frac{16\pi}{17}=\cos\frac{18\pi}{17}=-.9829730997$.

Note XXXIII.—Page 120.

Pythagoras was the first who remarked the simple property, that only three regular figures,—the square, the equilateral triangle and the hexagon,—can be constituted about a point. Here the mystic philosopher might again admire the union of the *monad* with the *triad*.—It may not be superfluous perhaps to observe, that on this property is founded the adaptation of patchwork, and the construction of tessellated pavement.

Note XXXIV.—Page 123.

The words, λογος in Greek and *ratio* in Latin, *signifying reason* or *manner of thought*, indicate vaguely a philosophical conception. The compound term αναλογια comes nearer to this idea; but its correlative, *proportio*, marks very distinctly a radical similarity of composition.

The doctrine of proportion has been a source of much controversy. In their mode of treating that important subject, authors differ widely; some rejecting the procedure of Euclid as circuitous and embarrassed, while others appear disposed to extol it as one of the happiest and most elaborate monuments of human ingenuity. But, to view the matter in its true light, we should endeavour previously to dispel that mist which has so long obscured our vision. The fifth book of Euclid, in its original form, is not found to answer the purpose of actual instruction; and this fact alone might justify a suspicion of its intrinsic excellence. The great object which the framer of the Elements had proposed to himself, by adopting such an artificial definition of proportion, was to obviate the difficulties arising from the consideration of incommensurable quantities. Under the shelter of a certain indefinitude of principle, he has contrived rather to evade those difficulties than fairly to meet them. Euclid seems not indeed to grasp the subject with a steady and comprehensive hold. In his seventh book, which treats of the properties of number, he abandons his former definition of proportion, for another that is more natural, though imperfectly developed. Through the whole contexture of the Elements, we may discern the influence of that mysticism which prevailed in the Platonic school. The language sometimes used in the fifth book would imply, that ratios are not mere conceptions of the mind, but have a real and substantial essence.

The obscurity that confessedly pervades the fifth book of Euclid being thus occasioned solely by the attempt to extend the definition of proportion to the case of incommensurables, the theory of which is contained in his tenth book—the pertinacity of modern editors of the Elements in retaining such an intricate definition, appears the more singular, since, omitting all the books relative to the properties of numbers, they have not given the slightest intimation respecting even the existence of incommensurable quantities.

The notion of proportionality involves in it necessarily the ideas of number. The doctrine of proportion hence constitutes a branch of universal arithmetic; and had I not on this occasion yielded to the prevalence of custom, I should have deferred the consideration of the subject till I came to treat of Algebra, where it is sometimes given, but in a very contracted form. The properties themselves are extremely simple, and may be regarded as only the exposition of the same principle under different aspects. The various transformations of which analogies are susceptible, exactly resemble the changes usually effected in the reduction of equations.

According to Euclid, "The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth." This definition, however perplexed and verbose, is yet easily derived from that which appears to furnish the simplest and most natural criterion of proportionality: For, let $A : B :: C : D$; it was stated as a fundamental principle, that, if the mth part of A be contained n times in B, the mth part of C will likewise be contained n times in D. Whence $nA = mB$, and $nC = mD$; which is the basis of Euclid's definition. But when the terms are incommensurable, such equality cannot *absolutely* subsist. In this case, no single trial would be sufficient for ascertaining proportionality. It is required that, *every* multiple whatever, mA, being greater or less than nB, the corresponding multiple, mC, shall

likewise be constantly greater or less than aD. Actually to apply the definition is therefore impossible; nor does it even assist us at all in directing our search. In the natural mode of proceeding, by assuming successively a smaller divisor, we are, at each time, brought nearer to the incommensurable limit. But Euclid's famous definition leaves us to grope at random after its object, and to seek our escape, by having recourse to some auxiliary train of reasoning or induction.

The author of the Elements has likewise given what Dr Barrow calls a *metaphysical* definition of ratio: "Ratio is a mutual relation of two *magnitudes* of the same kind to one another, in respect of *quantity*." This sentence, as it now stands, appears either tautological, or altogether void of meaning; and Dr Simson, anxious for the credit of Euclid, considers it, in his usual manner, as the interpolation of some unskilful editor. I am inclined to think, however, that the passage will admit of a version which is not only intelligible, but conveys a most correct idea of the nature of ratio. The original runs thus: *Λογος εστι δυο μεγεθῶν ὁμογενῶν ἡ κατα πηλικοτητα προς αλληλα ποια σχεσις.* Now the term *πηλικος*, on which the whole evidently hinges, though commonly rendered *quantus*, may be translated *quotus*, as expressing either *magnitude* or *multitude*. In its primitive sense, it probably denoted *number*, and came afterwards to signify *quantity*, as this word itself has, in the French language, undergone the reverse process. In confirmation of this opinion, it may be stated, that the relative term *ἡλικια* properly denotes *age*, and thence *stature* or *size*. According to this interpretation, therefore, "Ratio is a certain mutual habitude of two homogeneous magnitudes with respect to *quotity*, or numerical composition."

Note XXXV.—Page 134.

This proposition is easily derived from geometry; for, since of proportional lines the rectangle under the extremes is equal to that of the means, the segments AG and AH of the diameter in the figure of page 93 are (III. 7. El.) the greatest and least terms of an analogy, of which AB and AD are the intermediate terms, and consequently (III. 6. El.) the diameter GH, or the sum of AG and AH, is greater than the chord BD, or the sum of AB and AD.

Note XXXVI.—Page 143.

It is more convenient, however, to derive the numerical ratio, from the quotients of subdivision in their natural order; and this method has besides the peculiar advantage of exhibiting a succession of elegant approximations.

The quantities A, B, C, D, &c. are determined, as before, by these conditions: $A=mB+C$, $B=nC+D$, $C=pD+E$, $D=qE+F$, &c. But other expressions will arise from substitution: For,

1. $A=mB+C=m(nC+D)+C=(mn+1)\,C+mD$, or, putting $m.n+1=m'$, $A=m'C+mD$.

2. $A=m'C+mD=m'(pD+E)+mD=(m'p+m)\,D+m'E$, or, putting $m'.p+m=m''$, $A=m''D+m'E$.

3. $A=m''D+m'E=m''(qE+F)+m'E=(m''q+m')\,E+m''F$, or, putting $m''q+m'=m'''$, $A=m'''E+m''F$.

Again, the successive values of B are developed in the same manner:

1. $B=nC+D=n(pD+E)+D=(np+1)\,D+nE$, or, putting $n.p+1=n'$, $B=n'D+n.E$.

2. $B=n'D+nE=n'(qE+F)+nE=(n'q+n)\,E+n'F$, or, putting $n'.q+n=n''$, $B=n''E+n'F$.

These results will be more apparent in a tabular form:

$A=mB+C,$	$B=nC+D,$
$=m'C+mD,$	$=n'D+nE,$
$=m''D+m'E,$	$=n''E+n'F,$
$=m'''E+m''F,$	&c.
&c.	

The substitutions are thus arranged:

$m.n+1=m',$	$n.p+1=n',$
$m'.p+m=m'',$	$n'.q+n=n'',$
$m''.q+m'=m''',$	&c.
&c.	

Whence, the law of the formation of the successive quantities, is easily perceived.

But, to find the ratio of A to B, it is not requisite to know the values of the remainders C, D, E, &c. Suppose the subdivision to terminate at B; then $A = mB$, and consequently $A : B$, as $mB : B$, or $m : 1$. If the subdivision extend to C, then $A = m'C$, and $B = nC$; whence $A : B$, as $m' : n$. In general, therefore, the second term, in the expressions for A and B, may be rejected, and the letter which precedes it considered as the ultimate measure, and corresponding to the arithmetical unit. Hence, resuming the substitutions and combining the whole in one view, it follows, that the ratio of A to B may thus be successively represented:

1. $m : 1$.
2. $mn + 1 : n$, or $m' : n$.
3. $m'p + m : np + 1$, or $m'' : n'$.
4. $m''q + m' : n'q + n$, or $m''' : n''$.

&c. &c. &c.

The formation of these numbers will evidently stop, when the corresponding subdivision terminates. But even though the successive decomposition should never terminate, as in the case of incommensurable quantities,—yet the expression thus obtained must constantly approach to the ratio of A : B, since they suppose only the omission of the remainder of the last division, and which is perpetually diminishing.

Note XXXVII.—Page 144.

The same conclusion is derived from the division of surds. Thus $\frac{\sqrt{2}}{1} = 1 + \frac{\sqrt{2}-1}{1}$, $\frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1} = 2 + \frac{\sqrt{2}-1}{1}$, and then continually the expansion of the same residue $\frac{1}{\sqrt{2}-1}$, which therefore gives 2 as a repeated integral quotient. Hence m being 1 and n, p, q, r, &c. all equal to 2, the successive approximations are, by the last note, 1 : 1, 2 : 3, 5 : 7, 12 : 17, 29 : 41, 70 : 99, &c.

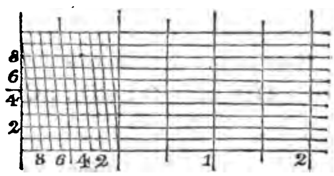
Note XXXVIII.—Page 150.

The consideration of diverging lines furnishes the simplest and readiest means, for transferring the doctrine of proportion to geometrical figures. The order which Euclid has followed, beginning with parallelograms, and thence passing from surfaces to lines, appears to be less natural.

Note XXXIX.—Page 153.

It will be proper here to notice the several methods adopted in practice, for the minute subdivision of lines. The earliest of these—the *diagonal scale*—depending immediately on the proposition in the text, is of the most extensive use, and constituted the first improvement on astronomical instruments.

Thus, in the figure annexed, the extreme portion of the horizontal line is divided into ten equal parts, each of which again is virtually subdivided into ten secondary parts. This subdivision is effected by means of diagonal lines, which decline from the perpendicular by intervals equal to the primary divisions, and which are cut transversely into ten equal segments by equidistant parallels. Suppose, for example, it were required to find the length of 2 and 38—100 parts of a division; place one foot of the



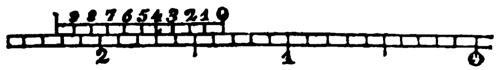
compasses in the second vertical at the eight interval which is marked with a dot, and extend the other foot, along the parallel, to the dot on the third diagonal. The distance between these dots may, however, express indifferently 2,38, 23.8, or 238, according to the assumed magnitude of the primary unit.

Nunez, or Nonius, proposed one more complicated. He placed a number of parallel scales, differently divided, and forming a regular descending gradation. An index laid any where across these scales would, therefore, cut at least one of them at some division, and hence the intercepted space would be expressed by a corresponding fraction.

But the method of subdivision afterwards introduced by Vernier, is much simpler and far more ingenious. It is founded on the difference of two approximating scales, one of which is moveable. Thus, if a space equal to $n-1$ parts on the limb of the instrument be divided into n parts, these evidently will each of them be smaller than the former, by the nth part of a division. Wherefore, on shifting forward this parasite scale, the quantity of aberration will diminish at each successive division, till a new coincidence obtains, and

then the number of those divisions on that scale will mark the fractional value of the displacement.

Thus in the annexed figure, nine divisions of the primary scale forming ten equal parts in the attached or sliding scale, the moveable zero stands beyond the first interval between the third and fourth division. To find this minute difference, observe where the opposite sections of the scales come to coincide, which occurs under the fourth division of the sliding scale, and therefore indicates the quantity 1.34.



Note XL.—Page 158.

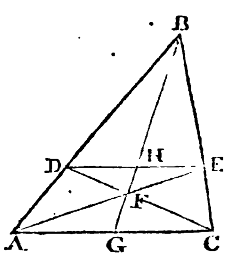
This corollary is easily deduced by a direct process; for CD : DE : : DE : OD, and CD : OD : : CD^2 : DE^2 or AD $\times$ DB.

Note XLI.—Page 160.

If two straight lines be inflected from the extremities of the base of a triangle to cut the opposite sides proportionally, another straight line, drawn from the vertex through their point of concourse, will bisect the base.

In the triangle ABC, let AE and CD, drawn from the extremities of the base to cut the opposite sides proportionally, intersect each other in F, join BF, which produce if necessary to meet the base in the point G; AG will be equal to GC.

For join DE. And because the sides AB and BC are cut proportionally, DE is parallel to AC (VI. 1. cor.), whence BD : BA : : BH : BG (VI. 1.); but BD : BA : : DE : AC (VI. 2.), and therefore BH : BG : : DE : AC. Again, the parallels DE and AC being cut by the diverging lines AE and CD, DE : AC : : DF : FC (VI. 2.), and DF : FC : : FH : FG (VI. 1.); wherefore BH : BG : : FH : FG, or BF is cut internally and externally in the same ratio. But



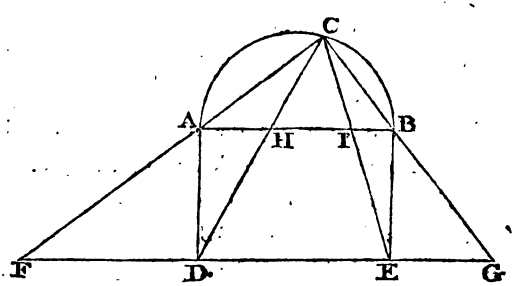
DH being parallel to AG, BH : BG : : DH : AG; and since DH is also parallel to GC, HF : FG : : DH : GC; whence DH : AG : : DH : GC, and consequently AG is equal to GC.

Note XLII.—Page 165.

If a semicircle be described on the side of a rectangle, and through its extremities two straight lines be drawn from any point in the circumference to meet the opposite side produced both ways; the altitude of the rectangle will be a mean proportional between the segments thus intercepted.

Let ABED be a rectangle, which has a semicircle ACB described on the side AB, and the straight lines CA and CB drawn from a point C in the circumference to meet the extension of the opposite side DE; the altitude AD of the rectangle will be a mean proportional between the exterior segments FD and EG.

For, the angle ADF, being evidently a right angle, is equal to the angle ACB, which stands in a semicircle (III. 22.), and the angle DFA is equal to the exterior angle BAC (I. 23.); wherefore (VI. 12.) the triangle FAD is similar to ABC. In the same manner, it is proved that the triangle BGE is similar to ABC; whence the triangles DFA and BGE are similar to each other, and consequently (VI. 12.) FD : AD : : BE or AD : EG.



If the straight lines CD and CE be drawn, they will (VI. 2.) divide the diameter AB into segments AH, HI, and IB, which are respectively proportional to the segments FD, DE, and EG of the extended side DE. Consequently when ABED is a square, and therefore DE a mean proportional between FD and EG, it must follow that HI is likewise a mean proportional between AH and IB.

If the rectangle ABED have its altitude AD equal to the side

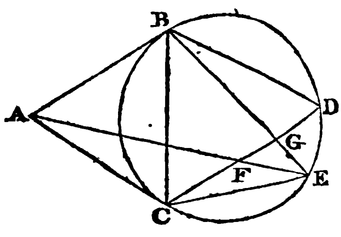
of a square inscribed within the circle, the square of the diameter AB is equivalent to the squares of the two segments AI and BH. For FD : AD : : AD : EG, whence (V. 6.) $FD.EG=AD^2$, or $2FD.EG=2AD^2$; but (IV. 16. cor.) $2AD^2=AB^2$ or DE^2, and consequently $2FD.EG=DE^2$; wherefore (VI. 2.) $2AH.IB=HI^2$, and, hence, by Note XXI. the segments AI, BH are the sides of a *right-angled* triangle, of which AB is the hypotenuse, or $AB^2=AI^2+BH^2$.

Note XLIII.—Page 166.

A chord of a circle is divided in continued proportion, by straight lines inflected to any point in the opposite circumference from the extremities of a parallel tangent, which is limited by another tangent applied at the origin of the chord.

Let AB, AC be two tangents applied to a circle, CD a chord drawn parallel to AB, and AE, BE straight lines inflected to a point E in the opposite circumference; then will the chord CD be cut in continued proportion at the points F and G, or CF : CG : : CG : CD.

For join BD, BC, and CE. Because the tangent AB is equal to AC (III. 32. cor. 2.), the angle ABC is equal to ACB (I. 11.); but ABC is equal to the angle BCD (I. 23.), and to the angle BDC (III. 25.); whence (VI. 12.) the triangles BAC and BDC are similar, and AB : BC : : BC : CD, and consequently (V. 6.) $BC^2=AB.CD$. Again, the triangles CBG and CBE are similar, for they have a common angle CBE, and the angle BCG,



or BCD, is equal to BDC, or BEC (III. 18.): Wherefore BG : BC : : BC : BE, and $BC^2=BG.BE$. Hence $AB.CD=BG.BE$, and AB : BE : : BG : CD; but FG being parallel to AB, AB : BE : : FG : GE (VI. 2.), and consequently FG : GE : : BG : CD; therefore (V. 6.) $FG.CD=BG.GE$; and since (III. 32.) $BG.GE=CG.GD$, it follows that $CG.GD=FG.CD$, and FG : CG : : GD : CD, and hence (V. 10.) CF : CG : : CG : CD.

Note XLIV.—Page 166.

The chord DG in the second construction is hence equal to the tangent in the third. But the tangent being at right angles to the radius GO, is less than DO; and therefore the geometrical, is less than the arithmetical, mean.—It may be observed, that, in geometrical constructions, the transition from the sine to the tangent frequently takes place, each of these lines being perpendicular to a limiting radius.

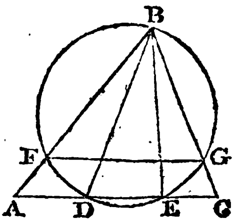
Note XLV.—Page 170.

This well-known proposition is now rendered more general, by its extension to the case of the exterior angle of the triangle. The two cases combined afford an easy demonstration of the corollary to Prop. 7. Book VI.; for the straight lines bisecting the vertical and its adjacent angle form a right-angled triangle, of which the hypotenuse is the distance on the base between the points of internal and external section.

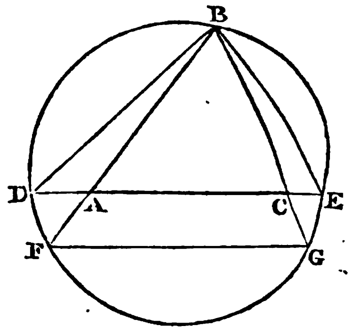
Note XLVI.—Page 170.

If, from the vertex of a triangle, two straight lines be drawn, making equal angles with the sides and cutting the base; the squares of the sides are proportional to the rectangles under the adjacent segments of the base.

In the triangle ABC, let the straight lines BD and BE make the angle ABD equal to CBE; then $AB^2 : BC^2 :: DA.AE : EC.CD$.

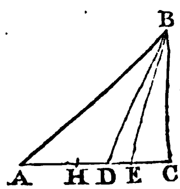


For (III. 10. cor.) through the points B, D, and E describe a circle, meeting the sides AB and BC of the triangle in F and G, and join FG.

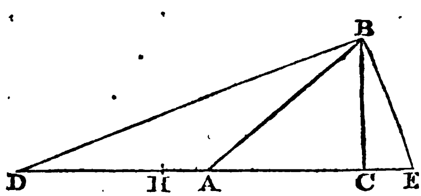


Because the angles DBF and EBG are equal, they stand (III. 18. cor.) on equal arcs DF and EG, and consequently (III. 20. cor.) FG is parallel to DE. Whence (VI. 1.) AB : BC : : AF : CG, and therefore (V. 13.) $AB^2 : BC^2 :: AB.AF : BC.CG$; but (III. 32.) $AB.AF = DA.AE$, and $BC.CG = EC.CD$. Wherefore $AB^2 : BC^2 :: DA.AE : EC.CD$.

If the triangle ABC be right-angled at C, and the vertical lines BD and BE cut the base internally; then $BC^2+AC.CE : BC^2 : : AE : CD$. For make AH equal to EC. Because $AB^2 : BC^2 :: DA.AE : EC.CD$, and (II. 11.) $AB^2 = AC^2+BC^2$, therefore $AC^2+BC^2 : BC^2 :: DA.AE : EC.CD$, and, by division, $AC^2 : BC^2 :: DA.AE-EC.CD : EC.CD$. But, by successive decomposition, $DA.AE-EC.CD = DA.AC - DA.EC - EC.CD = DA.AC - EC.AC = AC.HD$; whence $AC^2 : BC^2 :: AC.HD : EC.CD$, and (V. 13. and cor.) $AC.EC : BC^2 :: EC.HD : EC.CD$, or (V. 3.) $HD : CD$; consequently (V. 9.) $BC^2+AC.EC : BC^2 :: HC : CD$; but, AH being equal to EC, HC is equal to AE; wherefore $BC^2+AC.EC : BC^2 :: AE : CD$,

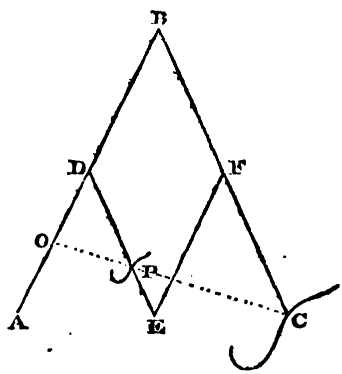


If the vertical lines BD, BE cut the base AC of a right-angled triangle ACB externally; then will $BC^2-AC.EC : BC^2 :: AE : CD$. For make AH=EC. It is demonstrated as before, that $AC^2 : BC^2 :: DA.AE-EC.CD : EC.CD$; but $DA.AE-EC.CD = DA.AC + DA.EC - EC.CD = DA.AC - EC.AC = AC.HD$; wherefore $AC^2 : BC^2 :: AC.HD : EC.CD$, and $AC.EC : BC^2 :: EC.HD : EC.CD :: HD : CD$, and consequently $BC^2-AC.EC : BC^2 ::$ HC or AE : CD.



Note XLVII.—Page 175.

The latter part of the scholium was added, with a view to explain the principle of the construction of the *pantagraph*, a very useful instrument contrived for copying, reducing, or even enlarging plans. It consists of a jointed rhombus DBFE, framed of wood or brass, and having the two sides BD and BF extended to double their length; the side DE and the branch DA are marked from D with successive divisions, DO being made to BO always in the ratio of DP to BC; small sliding boxes for holding a pencil or tracing point are brought to the corresponding graduations, and secured in their position by screws; the point O is made the centre of motion, and rests on a fulcrum or support of lead; and the tracer is generally fixed at C, while the crayon or drawing point is lodged at P. From the property of diverging lines intersecting parallels, the three points O, P and C must evidently range in the same straight line, and which is divided at P in the determinate ratio. While the point C, therefore, is carried along the boundaries of any figure, the intermediate point P will, by the scholium, trace out a similar figure, reduced in the proportion of OC to OP or of OB to OD, and which, in the present instance, is that of three to one.



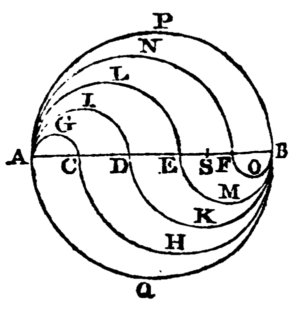
But the point P may be placed on the fulcrum, the tracer inserted at O, and the crayon held at C; in which case, C would delineate a figure which is enlarged in the ratio of OP to PC or of OD to DB. If the points O and P were now brought to coincide with A and E, the distances AE and EC being equal, the original figure would be transferred into a copy exactly of the same dimensions.

In reducing small figures, however, artists commonly prefer another method, which is partly mechanical. The original is divided into a number of small squares, by means of equidistant and inter-

secting parallels. Other reduced squares are drawn for the copy, which is then filled up, by observing the same relative position and form of the boundaries.—One material advantage results from this practice; for if oblongs be used in the copy instead of squares, the original figure will be more reduced in one dimension than another, which is often very convenient where height and distance are represented on different scales.

Note XLVIII.—Page 181.

The curious properties of the *crescents*, or *lunulæ*, *contained* in the first corollary, were discovered by Hippocrates of Chios, in his attempts to square the circle. But a beautiful extension of the same principle was briefly suggested by Mr Lawson, and afterwards explained and demonstrated in Dr Hutton's *Mathematical Tracts*. It is a mode of dividing a given circle into equal portions and contained within equal circular boundaries. For example, let it be required to cut the circle APBQ into five equal spaces: Divide the diameter AB into five equal parts at the points C, D, E, and F; on AC, AD, AE, and AF describe the semicircles AGC, AID, ALE, and ANF, and on BC, BD, BE, and BF, towards the opposite side, describe the semicircles BHC, BKD, BME, and BOF; the circle APBQ will be divided into five equal portions, by the equal compound semicircumferences AGCHB, AIDKB, ALEMB, and ANFOB.



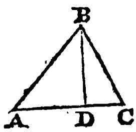
For the diameter AB is to the diameter AD, as the circumference of AB to the circumference of AD, or (V. 3.), as the semicircumference APB to the semicircumference AID; and AB is to BD, as the semicircumference APB to the semicircumference BKD. Wherefore (V. 20.) AB is to AD and BD together as the semicircumference APB to the compound boundary AIDKB; and consequently these interior boundaries AGCHB,

AIDKB, ALEMB, and ANFOB, are all equal to the semicircumference of the original circle.

Again, the circle on AB is to the circles on AE and AF, as the square of AB to the squares of AE and AF; and consequently (V. 20.) the circle on AB is to the difference between the circles on AE and AF, as the square of AB to the difference between the squares of AE and AF, that is (II. 19.), the rectangle under the sum and difference of AE and AF, or twice the rectangle under EF and AS, the distance of A from the middle point of EF. Whence the circle APBQ is to the difference of the semicircles ALE and ANF, or the space ALEFN, as the square of AB to the rectangle under AS and EF; and, for the same reason, the circle APBQ is to the space FOBME, as the square of AB is to the rectangle under BS and EF; consequently (V. 20.) the circle ABPQ is to the compound space ALEMBOFN, as the square of AB to the rectangles under AS and EF and BS and EF, or the rectangle under AB and EF; but the square of AB is to the rectangle under AB and EF, (V. 25. cor. 2.) as AB to EF, which is the fifth part of AB; wherefore (V. 5.) any of the intermediate spaces, such as ALEMBOFN, is the fifth part of the whole circle.

Note XLIX.—Page 183.

This elegant theorem admits of an algebraical investigation. Put AC$=a$, AB$=b$, BC$=c$, and let s denote the semiperimeter, and T the area of the triangle; then, by Prop. 26. Book II., $2\text{AC}\,.\,\text{CD} = a^2 + c^2 - b^2$, consequently



$\text{CD} = \frac{a^2+c^2-b}{2a}$, and $\text{BD}^2 = \text{BC}^2 - \text{CD}^2 =$

$c^2 - \left(\frac{a^2+c^2-b^2}{2a}\right)^2$, and, therefore, by

Prop. 6. Book II. $\text{T}^2 = \frac{\text{AC}^2.\text{BD}^2}{4} = \frac{4a^2c^2-(a^2+c^2-b^2)^2}{16}$.

But this expression, consisting of the difference of two squares, may be decomposed, by Prop. 19. Book II.; whence $\text{T}^2 =$

$$\frac{2ac+a^2+c^2-b^2}{4}\cdot\frac{2ac-a^2-c^2+b^2}{4} = \frac{(a+c)^2-b^2}{4}\cdot\frac{b^2-(a-c)^2}{4};$$

and, decomposing these factors again,

$$T^2 = \frac{a+b+c}{2} \cdot \frac{a-b+c}{2} \cdot \frac{a+b-c}{2} \cdot \frac{-a+b+c}{2}.$$

Now $\frac{a+b+c}{2} = s$, $\frac{a-b+c}{2} = s-b$, $\frac{a+b-c}{2} = s-c$,

and $\frac{-a+b+c}{2} = s-a$; wherefore we obtain, by substitution,

$$T = \sqrt{(s(s-a)(s-b)(s-c))}.$$

This most useful proposition was known to the Arabians, but seems to have been re-invented in Europe about the latter part of the fifteenth century.

Note L.—Page 186.

This ingenious and concise approximation to the quadrature of the circle was first published at Padua, in the year 1668, by my illustrious predecessor James Gregory. It is the more deserving of attention, as it seems to have led that original author to the invention of the method of series.

The Appendix to the books of Geometry cannot fail, by its novelty and singular beauty, to prove highly interesting. The first part is taken from a scarce tract of Schooten, who was professor of Mathematics at Leyden, early in the seventeenth century. But the second and most important part is chiefly selected from a most ingenious work of Mascheroni, a celebrated Italian mathematician, which in 1798 was translated into French, under the title of *Geometrie du Compas*. It will be perceived, however, that I have adapted the arrangement to my own views, and have demonstrated the propositions more strictly in the spirit of the ancient geometry.

Note LI.—Page 211.

These three books are designed to exhibit a distinct and comprehensive view of the mode by which the Greek geometers conducted their Analysis. For that purpose, I have chosen a series of propo-

sitions, at first extremely simple, but gradually rising in difficulty as the train of investigation proceeds. The first book, being rather of a miscellaneous nature, is drawn from a variety of sources. The 25th and 26th Propositions contain the different analyses of the two problems so famous in the Platonic school—*the trisection of an angle*—and *the duplication of the cube*—which led immediately to the cultivation of the higher geometry. The concluding theorem is the only one supplied by the *Data* of Euclid.

In the second and third books, I have endeavoured to comprise all that relates to the ancient analysis in its most improved state, as extended by the labours of Apollonius and his illustrious contemporaries. Without omitting any material proposition, I have yet avoided the prolixity of pursuing in detail their numerous subdivisions. Our system of modern education, embracing such a wide range, would scarcely indeed afford leisure for indulging in those easy tasks.

The method of analysis, so deservedly valued in the ancient schools, was regularly studied after the Elements of Geometry. According to Pappus, it consisted of eight distinct treatises:

1. The *Data*—περὶ τῶν δεδομένων—in a single book of considerable length, but containing propositions only of the very simplest kind.

2. The *Section of Ratio*—περὶ λογου ἀποτομῆς—in two books, which Dr Halley, with much sagacity and incredible labour, restored, from a MS. in the Bodleian library. The object of the tract was the solution of this problem, branched out into a multitude of cases, and marked with various limitations: "Through a given point to draw a straight line intercepting segments on two straight lines which are given in position, from given points and in a given ratio." It forms the first four propositions of the second book.

3. The *Section of Space*—περὶ χωριου ἀποτομῆς—in two books. Of these no vestige remained; but Dr Halley, guided by a few hints from Pappus, very successfully exerted his ingenuity in divining the original structure. It was proposed—"Through a given point, to draw a straight line cutting off segments from given points on two straight lines given in position, and which shall contain a rectangle equal to a given space." This occupies the propositions from the 5th to the 10th inclusively of the second book.

4. The *Determinate Section*—περὶ διωρισμένης τομῆς—in two books,

These were also lost; but Dr Simson, assisted by the attempts of Schooten, has restored them in the most luminous manner. Their object was—"To find a point, the rectangles or squares of whose distances from given points in the same straight line should have a determined ratio. They form Prop. 10—19. Book II.

5. *Inclinations*—περὶ νεύσεων—in two books. It was proposed—"To insert a straight line, of a given magnitude, and tending to a given point, between two lines which are given in position." This problem was restored by Marinus Ghetaldus, a patrician of Ragusa; and other investigations were given by Hugo de Omerique, in his ingenious treatise on Geometrical Analysis, printed at Cadiz in 1698. Two solutions of the case of the rhombus, remarkable for their elegance, appeared in the posthumous works of Huygens, who was imbued with the finest taste for the ancient geometry: I have condensed the whole in Prop. 19—26. Book II.

6. *Tangencies*—περὶ ἐπαφῶν—in two books. Of this tract only some lemmas were preserved, which enabled the celebrated Vieta in a great measure to restore it. Some of the cases which had escaped him were solved by Marinus Ghetaldus; and farther improvements were made in 1612, by Alexander Anderson of Aberdeen, an ancestor of the Gregorys. The general problem occupies the remainder of the second book.

7. *Plane Loci*—περὶ τόπων ἐπιπέδων—in two books. The object was—"To find the conditions under which a point, varying in its position, is yet confined to trace a straight line or a circle given in position." This beautiful train of investigation was partly restored by Schooten in 1650, though after a sort of algebraical form. The ingenious Fermat succeeded in bestowing greater simplicity on the subject. But all these attempts have been eclipsed by Dr Simson, whose treatise *De Locis Planis*, published at Glasgow in 1749, is a model of geometrical strictness and elegance. The first 16 propositions of the third Book include all the principal theorems, which I have selected with additions.

The six preceding branches of analysis were all the creation of Apollonius of Perga, the most assiduous and inventive of the Greek geometers.

8. *Porisms*—περὶ τῶν πορισμάτων—in three books, composed by

Euclid. No trace of these now remains, except some obscure hints of Pappus, rendered still more perplexed by the corrupt and mutilated state of his text. The subject had long proved an ænigma which it baffled the efforts of the ablest and most learned mathematicians to unravel. Fermat advanced some steps; but the honour of completing the discovery was reserved for our countryman Dr Simson, whose restoration of the Porisms was given to the scientific world in 1776, in a posthumous volume, printed at the expence of the late Earl Stanhope. From that work I have extracted what seemed the best suited to my purpose; and I have likewise availed myself of the judicious remarks and illustrations of my distinguished colleague, Professor Playfair. These porisms, with some additions, are contained in Prop. 18—25. Book III.

The remaining propositions of the third book relate to the subject of *Isoperimeters*; which I have treated with the conciseness of the moderns, without departing, I hope, from the spirit of the ancient geometry.

Note LII.—Page 241.

This proposition is only a very limited case of the general problem of *inclinations*, which occupies inclusively from the 19th to the 25th Propositions of the Second Book of Analysis. The construction given in the text is immediately deduced from the second solution of Prop. 25. Book II.

Note LIII.—Page 247.

This and the next problem, which has been ascribed to a response of the *Delian* oracle, called forth the powers of the ancient analysis, and transcending the limits of elementary construction, led to the discovery of some of the higher curves, and essentially contributed to the extension of geometrical science. For the trisection of an angle, Nicomedes proposed the *conchoid*, a curve of such a nature, that every straight line drawn from a given point called its *pole* has the same portion intercepted between the curve and a straight line given in position and termed the *directrix*. An elegant solution of the problem is given in Newton's *Universal Arithmetic*, by means of an hyperbola whose asymptotes form an angle of 120°.

Note LIV.—Page 250.

In this proposition, I have condensed and endeavoured to simplify the fine speculations of the Greeks, respecting the duplication of the cube. The first analysis is that given by Hero, in his *Mechanical Institutions;* and the variation of it was proposed by Philo of Byzantium. The second analysis of the problem was given by Nicomedes, and the third by Pappus of Alexandria. In the first and second modes, the solution may be performed by the conchoid; in the third, it is effected by the *cissoid* of Diocles, which is so constituted, that any straight line, drawn from its *cusp*, has an equal portion intercepted by the curve, and by the generating circle and the *directrix*. Menechmus solved the problem in two ways—either by combining two parabolas—or by combining a parabola with a rectangular hyperbola.

Note LV.—Page 252.

Since the angle BDF is half of the angle ABC, and DF : BF : : $R : tan$BDF, it follows that, $4R : tan\frac{1}{2}ABC :: (AB+BC)^2-AC^2$: area of the triangle, or, by decomposition, $R : tan\frac{1}{2}ABC :: \left(\frac{AB+BC+AC}{2}\right)\left(\frac{AB+BC-AC}{2}\right)$: area of the triangle. It hence follows that, assuming the former notation, $T=s(s-AC)tan\frac{1}{2}ABC$. The same property might also be deduced by comparing Prop. 31. Book VI. of the Elements with Prop. 12. of the Trigonometry.

In Prop. 20. Book I. of Geometrical Analysis, it may be observed, that the limit occurs when the points F and F′ coincide; in which case $HF=FK$, $HF^2=GE^2=AG.AH$, and consequently $AE+AF$, at its greatest contraction, is equal to $AG+AH+2\sqrt{AG.AH}$.

Note LVI.—Page 288.

This and the six preceding propositions include those cases of the problem of *inclinations* which admit of an elementary construction. The first solution is borrowed from the geometrical analysis of Hugo de Omerique, and the second from the posthumous works of the celebrated Huygens. To solve the general problem would require the application of the conchoid.

Note LVII.—Page 297.

The first solution of this problem is taken from Dr Simson's posthumous works. But the second investigation, which is obviously shorter and more elegant, was communicated to me by my respected pupil Mr Wildig of Liverpool, to whose ingenuity and accurate taste in geometrical science I am glad to have this opportunity of bearing testimony, and to whose judicious remarks this edition is indebted for various improvements, as it owes much of its typographical correctness to his obliging and very patient revision of the sheets.

Note LVIII.—Page 311.

This proposition, extended to points in different planes, furnishes a legitimate demonstration of the remarkable property of projected masses, which forms, in Newton's *Principia*, the fourth corollary to the laws of motion; namely, that of any system of bodies impressed with uniform and rectilineal motions, the centre of gravity either remains at rest or travels uniformly in a straight line.

Note LIX.—Page 323.

It is easily perceived from the mode of successive construction, that the centre of the circle which terminates this process, must likewise be the centre of gravity of the several points. This curious property is noticed in Huygen's elegant tract, entitled *Horologium Oscillatorium*, and furnishes another example of the application of the principle of the *conservatio virium vivarum*, which has such extensive influence in the mutual action of bodies.

Note LX.—Page 325.

The porismatic point D is the centre of gravity of particles of matter situate at A, B and C; for MN being any straight line drawn through D, the distance CG is equal to the combined distances AH and BI, and consequently the opposite efforts of those particles, to turn their plane, must, about the centre D, maintain every way an exact equipoise.

The proposition might easily be extended to any number of points in the same plane; but it is true universally, if the points only have a determined position. The writers on Statics, however, have commonly assumed, what they were not entitled to do, the existence of an individual centre of gravity. This fundamental property of matter is simply and elegantly demonstrated by the ingenious Boscovich, in his *Theoria Philosophiæ Naturalis*, a work of very great and original merit.

Note LXI.—Page 329.

The composition of this problem is readily derived from Note XXX.; for $CE.CF=CG^2+GH.GI=CG^2+GD^2=CD^2$.

Note LXII.—Page 337.

This problem was first proposed by Sir Isaac Newton, for determining the path of a comet, from four observations made at given short intervals of time. But unfortunately it was afterwards found in practice to give uncertain or even erroneous results. This unexpected failure led Boscovich to examine closely the circumstances which might affect the solution, and he discovered that the problem becomes indeterminate or porismatic, in the very case where *its* aid is wanted to guide astronomical observation.

Note LXIII.—Page 343.

All the comparisons in geometry being originally founded on the properties of the triangle, and thence transferred to other rectilineal figures, it is evident that we can never reason directly concerning the circle, which can only be viewed in that respect as a polygon having innumerable sides. The consideration of limits, more or less disguised, must therefore unavoidably enter into every investigation which has for its object the mensuration of the circle.

Note LXIV.—Page 350.

The French philosophers have, at the instance of Borda, lately proposed and adopted the centesimal division of the quadrant, as easier, more consistent, and better adapted to our scale of arithme-

tic. On that basis, they have also constructed their ingenious system of measures. The distance of the Pole from the Equator was determined with the most scrupulous accuracy, by a chain of triangles extending from Calais to Barcelona, and since prolonged to the Balearic Isles. Of this quadrantal arc, the ten millionth part, or the tenth part of a second, and equal to 39.371 English inches, constitutes the *metre*, or unit of linear extension. From the metre again, are derived the several measures of surface and of capacity; and water, at its greatest degree of contraction, furnishes the standard of weights.

It would be most desirable, if this elegant and universal system were adopted, at least in books of science. Whether, with all its advantages, it be ever destined to obtain a general currency in the ordinary affairs of life, seems extremely questionable. At all events, its reception must necessarily be very slow and gradual; and, in the meantime, this innovation is productive of much inconvenience, since it not only deranges our habits, but tends to displace our delicate instruments and elaborate tables. The fate of the centesimal division may finally depend on the continued merit of the works framed after that model.

Note LXV.—Page 351.

The remarks contained in the preliminary scholium, will obviate an objection which may be made against the succeeding demonstrations, that they are not strictly applicable, except when the arcs themselves are each less than a quadrant. But this in fact is the only case absolutely wanted, all the derivative arcs being at once comprehended under the definition of the sine or tangent. To follow out the various combinations, would require a fatiguing multiplicity of diagrams; and such labour would still be quite superfluous, because the mode of extending or accommodating the results from the general principle is so easily perceived.

Note LXVI.—Page 356.

The general properties of the sines of compound arcs may be derived with great facility from Prop. 22. of Book VI. of the

Elements. For, since AB.CD+BC.AD=AC.BD, it is evident that $\frac{1}{2}$AB.$\frac{1}{2}$CD+$\frac{1}{2}$BC.$\frac{1}{2}$AD=$\frac{1}{2}$AC.$\frac{1}{2}$BD; but (cor. 1. def. Trig.) the semichord of an arc is the same as the sine of half the arc, and consequently, by substitution, $\sin\frac{1}{2}AB\sin\frac{1}{2}CD+\sin\frac{1}{2}BC\sin\frac{1}{2}ABCD=\sin\frac{1}{2}ABC\times\sin\frac{1}{2}BCD$. Let $\frac{1}{2}AB=L$, $\frac{1}{2}BC=M$, and $\frac{1}{2}CD=N$; wherefore $\frac{1}{2}ABCD=L+M+N$, $\frac{1}{2}ABC=L+M$, and $\frac{1}{2}BCD=M+N$, and hence the general result; $\sin L\sin N+\sin M\sin(L+M+N)=\sin(L+M)\sin(M+N)$, in which L, M and N are any arcs whatever. This expression, variously transformed, will exhibit all the theorems respecting sines. For the sake of conciseness, let the radius be denoted as usual by 1, and the semicircumference by π.

1. Put A=M, B=N, and let L be the complement of A. Then,

$$\cos A\sin B+\sin A\sin(A+B+\frac{\pi}{2}-A)=\sin(\frac{\pi}{2}-A+A)\sin(A+B);$$

that is, since the sine of an arc increased by a quadrant is the same as its cosine, $\sin A\cos B+\cos A\sin B=\sin(A+B)$.

2. Let the arc B be taken on the opposite side, or substitute $-B$ for it in the last expression, and $\sin A\cos B-\cos A\sin B=\sin(A-B)$.

3. In art 1, for A substitute its complement; then $\sin(A+B)=$

$\sin(\frac{\pi}{2}-A+B)=\sin(\frac{\pi}{2}+A-B)=\cos(A-B)$, and hence $\cos A\cos B+\sin A\sin B=\cos(A-B)$.

4. In art 2, likewise substitute for A its complement, and the result will become $\cos A\cos B-\sin A\sin B=\cos(A+B.)$

5. In art. 1, let A=B, and $2\sin A\cos A=\sin 2A$.

6. In art. 4, let A=B, and $\cos A^2-\sin A^2=\cos 2A$.

7. In art. 3, let A=B, and $\cos A^2+\sin A^2=1$.

8. Add the *formulæ* in art. 1 and 2, and $2\sin A\cos B=\sin(A+B)+\sin(A-B)$.

9. Subtract the *formulæ* of art. 2. from that of art. 1, and $2\cos A\sin B=\sin(A+B)-\sin(A-B)$.

10. Conjoin the *formulæ* of art. 3 and 4, and $2\cos A\cos B=\cos(A+B)+\cos(A-B)$.

11. Take the *formulæ* of art. 4. from that of art. 3, and $2.\sin A\sin B=\cos(A-B)-\cos(A+B)$.

12. In art. 8, let B be the complement of A, and $2\sin A^2=$

$\sin(A+\frac{\pi}{2}-A)+\sin(A-\frac{\pi}{2}+A)=1-\cos 2A=\text{vers}\,2A$.

13. In art. 9, let B be the complement of A, and $2cosA^2 =$ $sin(A+\frac{\pi}{2}-A)-(sinA-\frac{\pi}{2}+A)=1+cos2A=suvers2A$.

14. In art. 5, instead of A substitute its half, and $2sin\frac{1}{2}A \times cos\frac{1}{2}A =$ $sinA$.

15. In art. 6, likewise substitute the half of A for A, and $(cos\frac{1}{2}A)^2-(sin\frac{1}{2}A)^2=cosA$.

16. In art. 12, for A substitute its half, and $2(sin\frac{1}{2}A)^2=1-cosA$, or $sin\frac{1}{2}A=\sqrt{(\frac{1}{2}(1-cosA))}=\sqrt{\frac{1}{2}versA}$.

17. Make the same substitution in art. 13, and $2(cos\frac{1}{2}A)^2=$ $1+cosA$, or $cos\frac{1}{2}A=\sqrt{(\frac{1}{2}(1+cosA))}=\sqrt{\frac{1}{2}suversA}$.

18. In art. 8, transform A and B into $A+B$ and $A-B$, and consequently, for $A+B$ and $A-B$, substitute $2A$ and $2B$; then $2sin(A+B)cos(A-B)=sin2A+sin2B$, or $sin(A+B)cos(A-B)=$ $\frac{1}{2}(sin2A+sin2B)$.

19. Make the same transformation in art. 9, and $2cos(A+B)$ $sin(A-B)=sin2A-sin2B$, or $cos(A+B)sin(A-B)=$ $\frac{1}{2}(sin2A-sin2B)$.

20. Repeat this transformation in art. 10, and $2cos(A+B)$ $cos(A-B)=cos2A+cos2B$, or $cos(A+B)cos(A-B)=$ $\frac{1}{2}(cos2A+cos2B)$.

21. The same transformation being still made in art. 11, $2sin(A+B)sin(A-B)=cos2B-cos2A$, or $sin(A+B)sin(A-B)=$ $\frac{1}{2}(cos2B-cos2A)$.

22. Suppose $L=N=B$, and $M=A-B$; then the general expression becomes $sinB^2+sin(A-B)\ sin(A+B)=sinA^2$, or $sin(A+B)$ $sin(A-B)=sinA^2-sinB^2$.

23. Instead of A in the last article, take its complement, and $sin(\frac{\pi}{2}-A+B)sin(\frac{\pi}{2}-A-B)=cosA^2-sinB^2$, or $cos(A-B)$ $cos(A+B)=cosA^2-sinB^2$.

24. Compare art. 21. with 22, and $\frac{1}{2}(cos2B-cos2A)=sinA^2-$ $sinB^2$.

25. Comparing likewise art. 20. with 23, and $\frac{1}{2}(cos2A+cos2B)=$ $cosA^2-sinB^2$.

26. Resolve the difference of the squares in art. 22. into its factors, and $sin(A+B)sin(A-B)=(sinA+sinB)\ (sinA-sinB)$.

27. Make a similar decomposition in art. 23, and $\cos(A+B)\cos(A-B)=(\cos A+\sin B)(\cos A-\sin B)$.

24. In art. 18, instead of A and B take their halves, and $\sin A+\sin B=2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

25. Make the same change in art. 19, and $\sin A-\sin B=2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)$.

26. Change likewise art. 20, and $\cos B+\cos A=2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

27. Do the same thing in art. 21, and $\cos B-\cos A=2\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(A+B)$.

From Note XXVII. a very simple expression may be derived for the sum of the sines of progressive arcs. Suppose the diameter AO were drawn; then $BE+CF+DG=HG=HO+DO$, or $2\sin AB+2\sin AC+2\sin AD=HO+\sin AD$, and $\sin AB+\sin AC+\sin AD=\frac{1}{2}HO+\frac{1}{2}\sin AD=\frac{1}{2}AO.\tan BAO+\frac{1}{2}\sin AD$. Wherefore, in general, $\sin a+\sin 2a+\sin 3a\ldots\ldots\sin na=\frac{1}{2}\text{vers}\, na.\cot\frac{1}{2}a+\frac{1}{2}\sin na$. Hence the sum of the sines in the whole semicircle is $=\cot\frac{1}{2}a$. Thus, if the sines for each degree up to 180°, the radius being unit, were added together, the amount would be 114,58866.

Note LXVII.—Page 358.

On examining the formation of the successive terms of the first and second tables, it will appear that the coefficients are certain multiples of the powers of 2, whose exponents likewise at every step decrease by two. It is farther manifest that if 1, A, B, C, &c. 1, A′, B′, C′, &c. and 1, A‵, B‵, C‵, &c. denote the multiples corresponding to the arcs $n.a$, $\overline{n+1}.a$, and $\overline{n-1}.a$; then $A+1=A'$, $B+A‵=B'$, $C+B‵=C'$, &c. Whence the values of A, B, C, &c. are determined, either by the method of finite differences, adopting the appropriate notation, or from the theory of functions. Thus, in the first table, $\Delta A=1$, and $A=n-2$; $\Delta B=A‵=n-3$, and $B=\frac{\overline{n-3}\,\overline{n-4}}{2}$; $\Delta C=B‵=\frac{\overline{n-4}\,\overline{n-5}}{2}$, and $C=\frac{\overline{n-4}.\overline{n-5}.\overline{n-6}.}{2.3}$. Wherefore in general

$$(1.)\ Sin\ na = 2^{n-1}.\ c^{n-1}s - n-2.2^{n-3}c^{n-3}s + \frac{n-3.n-4}{2}.\ 2^{n-5}c^{n-5}s -$$

$$\frac{n-4.n-5.n-6}{2.3}.\ 2^{n-7}c^{n-7}s + \&c.$$

$$(2.)\ Cos\ na = 2^{n-1}.c^{n} - n.2^{n-3}.c^{n-2} + \frac{n.n-3}{2}.\ 2^{n-5}c^{n-4} -$$

$$\frac{n.n-4.n-5}{2.3}.\ 2^{n-7}.c^{n-6} + \&c.$$

The third and fourth tables are evidently formed by multiplying constantly by $2cos\ 2a$ or $2-4s^2$, and subtracting the term preceding; or the multiplication by $4s^2$ produces the second differences of the successive quantities. Hence in the former, $\Delta\Delta A = 4n''$, $\Delta\Delta B = 4A''$, &c.;

wherefore $\Delta\, A = n+1.n+1$, and $A = \frac{n.n-1.n+1}{2.3}$;

$\Delta B = \Sigma(\frac{2.n+2.n+1.n+3}{3}) = \frac{n+1.n+1.n-1.n+3}{3.4}$,

and $B = \frac{n.n-1.n+1.n-3.n+3}{2.3.4.5}$. But in the fourth table, $\Delta\Delta A = 4$, $\Delta\Delta B = 4A''$, $\Delta\Delta C' = 4B''$; and consequently $\Delta A = 2n+2$, and $A = \frac{n^2}{2}$; $\Delta B = \Sigma(2.n+2.n+2) = \frac{n.n+1.n+2}{3}$, and $B = \frac{n^2.n-2.n+2}{2.3.4}$. Wherefore in general,

$$(3.)\ Sin\ na = n.s - n.\frac{n^2-1}{2.3}s^3 + n.\frac{n^2-1}{2.3}.\frac{n^2-9}{4.5}s^5 -$$

$$n.\frac{n^2-1}{2.3}.\frac{n^2-9}{4.5}.\frac{n^2-25}{6.7}s^7 + \&c.$$

$$(4.)\ Cos\ na = 1 - \frac{n^2}{2}s^2 + \frac{n^2}{2}.\frac{n^2-4}{3.4}s^4 - \frac{n^2}{2}.\frac{n^2-4}{3.4}.\frac{n^2-16}{5.6}s^6 +, \&c.$$

In the fifth and sixth tables, the coefficients are evidently the same as those of the power of a binomial, only proceeding from both extremes to the middle terms. Hence, according as n is odd or even,

$$(5.)\ 2^{n-1}\ sin\ a^n = \pm sin\ na \mp n.\ sin(n-2)a \pm n.\frac{n-1}{2}sin(n-4)a \mp$$

$$n.\frac{n-1}{2}.\frac{n-2}{3}\ sin(n-6)a \pm \&c.;\ \text{and}$$

$$2^{n-1} \sin a^n = \pm \cos na \mp n.\cos(n-2)a \pm n.\frac{n-1}{2}\cos(n-4)a \mp$$

$$n.\frac{n-1}{2}.\frac{n-2}{3}\cos(n-6)a, \text{ \&c.}$$

Again,

(6.) $$2^{n-1} \cos a^n = \cos na + n.\cos(n-2)a + n.\frac{n-1}{2}.\cos(n-4a) +$$

$$n.\frac{n-1}{2}.\frac{n-2}{3}.\cos(n-6)a, \text{ \&c.}$$

In these three expressions, half the last term, which corresponds to the middle in the expansion of the binomial, is to be taken, when n is an even number.

It will be satisfactory likewise to subjoin an investigation of the sine of the multiple arc, as derived from the Theory of Functions.

It appears from inspecting the successive formation of the sines of the multiple arcs, 1. that the odd powers only of s occur; 2. that the coefficient of the first term is only n, and the other coefficients are its functions of third, fifth, &c. orders; and 3. that since, in the case when $n=1$, the rest of the coefficients evidently vanish, those coefficients in general, as affected by opposite signs, must in each term produce a mutual balance.

Let therefore $\sin nu = \overset{\prime}{n}.s + \overset{\prime\prime\prime}{n}.s^3 + \overset{\prime\prime\prime\prime\prime}{n}.s^5$ &c.; where s denotes the sine of the arc a, and $\overset{\prime}{n}$, $\overset{\prime\prime\prime}{n}$, $\overset{\prime\prime\prime\prime\prime}{n}$, &c. the successive odd orders of the functions of n. It is evident, from (Prop. 3. cor. 2. Trig.) that, by substitution

$$\left(\overset{\prime}{(n+1)} + \overset{\prime}{(n-1)}\right)s + \left(\overset{\prime\prime\prime}{(n+1)} + \overset{\prime\prime\prime}{(n-1)}\right)s^3 + \left(\overset{\prime\prime\prime\prime\prime}{(n+1)} + \overset{\prime\prime\prime\prime\prime}{(n-1)}\right)s^5 + \text{ \&c.}$$

$$= 2\sqrt{(1-s^2)}\ \sin na = (2 - s^2 - \tfrac{1}{4}s^4, \text{ \&c.})\ (\overset{\prime}{n}s + \overset{\prime\prime\prime}{n}s^3 + \overset{\prime\prime\prime\prime\prime}{n}s^5, \text{ \&c.})$$

$= 2\overset{\prime}{n}s + (2\overset{\prime\prime\prime}{n} - \overset{\prime}{n})s^3 + (2\overset{\prime\prime\prime\prime\prime}{n} - \overset{\prime\prime\prime}{n} - \tfrac{1}{4}\overset{\prime}{n})s^5$, &c. Now, equating corresponding terms, and rejecting the powers of s, we obtain these general results:

$$2n' = 2n';\ \overset{\prime\prime\prime}{(n+1)} + \overset{\prime\prime\prime}{(n+1)} = 2\overset{\prime\prime\prime}{n} - \overset{\prime}{n};\ \overset{\prime\prime\prime\prime\prime}{(n+1)} + \overset{\prime\prime\prime\prime\prime}{(n-1)} = 2\overset{\prime\prime\prime\prime\prime}{n} - \overset{\prime\prime\prime}{n} - \tfrac{1}{4}\overset{\prime}{n}.$$

It remains hence to discover the several orders of the functions of n.

1. The equation $2n'=2n'$ contains a mere identical proposition; but other considerations indicate that n must always denote the first term, or that the first function of n is n itself.

2. The equation $\overset{\prime\prime\prime}{(n+1)}+\overset{\prime\prime\prime}{(n-1)}=2\overset{\prime\prime\prime}{n}-\overset{\prime}{n}$ fixes the conditions of the third function of n, which, from the nature of the relation, is obviously imperfect, and wants the second term. Put, therefore, $n'''=\alpha n^3+\beta n$; and, by substitution, $2\alpha n^3+6\alpha n+2\beta n=2\alpha n^3+2\beta n-n$. Equating now the corresponding terms, and $6\alpha=-1$, or $\alpha=-\frac{1}{6}$; but $\alpha+\beta=0$, and therefore $\beta=+\frac{1}{6}$.

Whence $\overset{\prime\prime\prime}{n}=-\frac{1}{6}n^3+\frac{1}{6}n=-n.\frac{n^2-1}{2.3}$.

3. Again, in the third equation, $\overset{\prime\prime\prime\prime\prime}{(n+1)}+\overset{\prime\prime\prime\prime\prime}{(n-1)}=2\overset{\prime\prime\prime\prime\prime}{n}-\overset{\prime\prime\prime}{n}-\frac{1}{4}n'$, substitute $\overset{\prime\prime\prime\prime\prime}{n}=\alpha n^5+\beta n^3+\gamma n$, and the conditions of the fifth order of the function of n will be determined by this compound expression: $2\alpha n^5+(20\alpha+2\beta)n^3+(10\alpha+6\beta+2\gamma)n=2\alpha n^5+(2\beta+\frac{1}{6})n^3+(2\gamma-\frac{1}{6}-\frac{1}{4})\,n$. Equate the corresponding terms, and $20\alpha+2\beta=2\beta+\frac{1}{6}$, or $\alpha=\frac{1}{120}=\frac{1}{2.3.4.5}$. In like manner, $10\alpha+6\beta+2\gamma=2\gamma-\frac{1}{6}-\frac{1}{4}$, and $\beta=-\frac{1}{36}-\frac{1}{24}-\frac{1}{72}=-\frac{1}{12}=\frac{-10}{2.3.4.5}$; but $\alpha+\beta+\gamma=0$, whence $\gamma=\frac{9}{2.3.4.5}$ . . . .

Collectively, therefore, $\overset{\prime\prime\prime\prime\prime}{n}=\frac{n^5-10n^3+9n}{2.3.4.5}=n.\frac{n^2-1}{2.3}.\frac{n^2-9}{4.5}$

Whence, resuming all the terms, $\sin na=ns-n.\frac{n^2-1}{2.3}s^3+n.\frac{n^2-1}{2.3}.\frac{n^2-9}{4.5}s^5-$&c. as before.

From the expression for the sine of a multiple arc, may be deduced the series for the sine of any arc, in terms of the arc itself, and conversely. Let $na=A$, and therefore $a=\frac{A}{n}$; if n be supposed indefinitely great, then a must be indefinitely small, and consequently in a ratio of equality to s. Whence, substituting

A for na, and $\frac{A}{n}$ for s in the general expression, there results,

$$\sin A = A - \frac{n^2-1}{2.3}\cdot\frac{A^3}{n^2} + \frac{n^2-1}{2.3}\cdot\frac{n^2-9}{4.5}\cdot\frac{A^5}{n^4} - \&c.$$

But n being indefinitely great, the composite fractions $\frac{n^2-1}{n^2}$ $\frac{n^2-9}{n^2}$, &c. are each in effect equal to unit, which forms their extreme limit. Consequently, assuming that modification,

$$\sin A = A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} -, \&c.$$

Again, putting $a=A$ and $s=S$, suppose n to be indefinitely small, and $\sin na = na = nA$; whence, by substitution,

$$nA = nS - n.\frac{n^2-1}{2.3}S^3 + n.\frac{n^2-1}{2.3}\cdot\frac{n^2-9}{4.5}S^5 -, \&c. \text{ and}$$

$$A = S - \frac{n^2-1}{2.3}S^3 + n.\frac{n^2-1}{2.3}\cdot\frac{n^2-9}{4.5}S^5 - \&c.$$

But, if n vanish from all the terms, the series will pass into a simpler form.

$$A = S + \frac{1}{2.3}S + \frac{1.9}{2.3.4.5}S^5 + \frac{1.9.25}{2.3.4.5.6.7}S^7 +, \&c.$$

By a similar investigation, the series for the cosine of an arc is likewise found.

$$\cos A = 1 - \frac{A^2}{1.2} + \frac{A^4}{2.3.4} - \frac{A^6}{2.3.4.5.6} +, \&c.$$

These series' are very commodious for the calculation of sines, since they converge with sufficient rapidity when the arc is not a large portion of the quadrant. Though the method explained in the text is on the whole much simpler, yet as the errors of computation are thereby unavoidably accumulated, it would be proper at intervals to calculate certain of the sines by an independent process.

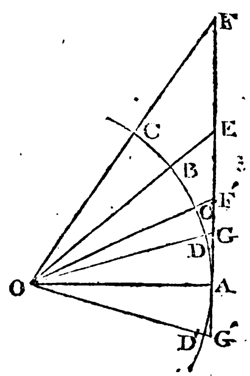
The series' now given furnish also various modes for the rectification of the circle. Thus, assuming an arc equal to the radius, its sine is, $1 - \frac{1}{2.3} + \frac{1}{2.3.4.5} - \&c. = .841471$, and its cosine is, $1 - \frac{1}{2} + \frac{1}{2.3.4} - \&c. = .440302$. But that arc evidently approaches

to 60°, of which the sine is $\sqrt{\frac{3}{4}}=.866025$, and the cosine .500000. Wherefore (Pr. 1. Trig.) the sine of the difference of these two arcs is $.866025\times.540302-.841471\times.500000=.04718$, and consequently, by the series, that interval itself is .0472. Hence the length of the arc of 60° is 1.0472, and the circumference of a circle which has unit for its diameter is $3\times1.0472=3.1416$; an approximation extremely commodious.

Note LXVIII.—Page 360.

This may be otherwise demonstrated from the corollaries to the proposition contained in Note XLVI.

Let AB and BC, or BC′, be two arcs, of which AB is the greater; make AD, or AD′, equal to BC, and apply the respective tangents. Because OAE is a right-angled triangle, and OG′, OF, are drawn, making equal angles with OA and OE, it follows, that $OA^2-AE.AG' : OA^2 :: EG' : AF$; and consequently $R^2-tanAB.tanBC : R^2 :: tanAB+tanBC : tan(AB+BC)$. Again, since OG and OF′ make equal angles with OA and OE, it is evident, that $OA^2+AE.AG : OA^2 :: EG : AF'$, and hence $R^2+tanAB\ tanBC : R^2 :: tanAB-tanBC : tan(AB-BC)$.



Note LXIX.—Page 360.

The radius being expressed by unit, the sum of the tangents of the angles of any triangle, is equal to the number arising from their continued product. For, let A, B, and C, denote the several angles of the triangle; and since two of these, such as A and B, are supplementary to the remaining one C, the tangent of A+B is the same (schol. def. Trig.) as that of the third angle in an opposite direction. Whence $\frac{tanA+tanB}{1-tanA.tanB}=-tan\,C$, and therefore $tan\,A+tan\,B=-tanC+tanA\,tanB\,tanC$, or $tanA+tanB+tanC=tanA\,tanB\,tanC$.

Note LXX.—Page 360.

The properties of the tangents are easily derived from those of the sines.

1. $TanA+tanB=\frac{sinA}{cosA}+\frac{sinB}{cosB}=\frac{sinAcosB+cosAsinB}{cosAcosB}=$ (art. 1. Note LXVI.) $\frac{sin(A+B)}{cosAcosB}$.

2. Change the sign of B in the last article, and $tanA-tanB=\frac{sin(A-B)}{cosAcosB}$.

3. Instead of A and B in art. 1. substitute their complements and $cotA+cotB=\frac{sin(A+B)}{sinAsinB}$.

4. Make the same substitution in art. 2, and $cotB-cotA=\frac{sin(A-B)}{sinAsinB}$.

5. $Tan(A+B)=\frac{sin(A+B)}{cos(A+B)}=$(art. 1. and 4. Note LXVI.) $\frac{sinAcosB+cosAsinB}{cosAcosB-sinAsinB}$, which, being divided by $cosA\,cosB$ or $sinA\,sinB$, gives $tan(A+B)=\frac{tanA+tanB}{1-tanA\,tanB}=\frac{cotB+cotA}{cotB\,cotA-1}$.

6. Change the sign of B in the last article, and $tan(A-B)=\frac{tanA-tanB}{1+tanA\,tanB}=\frac{cotB-cotA}{cotB\,cotA+1}$.

7. Divide the expression in the first article by that in the second, and $\frac{sin(A+B)}{sin(A-B)}=\frac{tanA+tanB}{tanA-tanB}=\frac{cotB+cotA}{cotB-cotA}$.

8. In the last article, change the sign of B, and instead of A take its complement, and $\frac{cos(A+B)}{cos(A-B)}=\frac{cotB-tanA}{cotB+tanA}=\frac{cotA-tanB}{cotA+tanB}$.

9. Divide the expression of art. 12. in Note LXVI. by that of art. 5., and $\frac{1-cos2A}{sin2A}=\frac{2sinA^2}{2sinAcosA}=\frac{sinA}{cosA}=tanA$.

10. Divide the expression of art. 5. in the same Note, by that of art. 13., and $\frac{sin2A}{1+cos2A}=\frac{2sinAcosA}{2cosA^2}=\frac{sinA}{cosA}=tanA$.

11. Multiply the expressions of the two preceding articles, and $\frac{1-cos2A}{1+cos2A}=tanA^2$, or $tanA=\sqrt{\frac{1-cos2A}{1+cos2A}}$.

12. Decompose the expression in art. 9., and $tanA=\frac{1}{sin2A}-\frac{cos2A}{sin2A}$ $=cosec2A-cot2A$.

13. In the last article, change A into its complement, and $cotA=$ $cosec2A+cot2A$.

14. Subtract the last expression from the one preceding it, and $tanA-cotA=-2cot2A$, or $tanA=cotA-2cot2A$.

15. In art. 9. 10. and 11. for 2A and A, take A and ½A, and $tan\frac{1}{2}A=\frac{1-cosA}{sinA}=\frac{sinA}{1+cosA}=\sqrt{\frac{1-cosA}{1+cosA}}$.

16. Multiply the expressions of art. 1. and 2., and $(tanA+tanB)$ $(tanA-tanB)=tanA^2-tanB^2=\frac{sin(A+B)\,sin(A-B)}{cosA^2\,cosB^2}$.

17. Multiply the expressions of art. 3. and 4., and $(cotB+cotA)$ $(cotB-cotA)=cotB^2-cotA^2=\frac{sin(A-B)\,sin(A+B)}{sinA^2\,sinB^2}$.

18. Divide art. 28. of Note LXVI. by art. 29., and $\frac{sinA+sinB}{sinA-sinB}=$ $\frac{2sin\frac{1}{2}(A+B)\,cos\frac{1}{2}(A-B)}{2cos\frac{1}{2}(A+B)\,sin\frac{1}{2}(A-B)}=\frac{tan\frac{1}{2}(A+B)}{tan\frac{1}{2}(A-B)}$.

19. Divide art. 30. of the same Note by art. 31., and $\frac{cosB+cosA}{cosB-cosA}=$ $\frac{2cos\frac{1}{2}(A+B)\,cos\frac{1}{2}(A-B)}{2sin\frac{1}{2}(A+B)\,sin\frac{1}{2}(A-B)}=\frac{cot\frac{1}{2}(A+B)}{tan\frac{1}{2}(A-B)}$.

Since by art. 14. $cotA-2cot2A=tanA$, if the arc A and its compound expression be continually bisected, there will arise:

$$\tfrac{1}{2}cot\tfrac{1}{2}A-\ cot\ A=\tfrac{1}{2}tan\tfrac{1}{2}A$$
$$\tfrac{1}{4}cot\tfrac{1}{4}A-\tfrac{1}{2}cot\tfrac{1}{2}A=\tfrac{1}{4}tan\tfrac{1}{4}A$$
$$\tfrac{1}{8}cot\tfrac{1}{8}A-\tfrac{1}{4}cot\tfrac{1}{4}A=\tfrac{1}{8}tan\tfrac{1}{8}A$$

&c. &c. &c.

Wherefore, collecting these successive terms, and observing the effects of the opposite signs, the general result will come out,

$$\frac{1}{2^n}cot\frac{A}{2^n}-cotA=\tfrac{1}{2}tan\tfrac{1}{2}A+\tfrac{1}{4}tan\tfrac{1}{4}A+\tfrac{1}{8}tan\tfrac{1}{8}A\ \ldots\ +\frac{1}{2^n}tan\frac{A}{2^n}.$$

exceeds only by 3 in the last place the logarithm of 3,141592654. As the successive terms come to form very nearly a progression that descends by quotients of 4, the third of the last one is, for the reason stated in page 363, considered as equal to the result of the continued addition.

Note LXXII.—Page 367.

An elegant mode of forming the approximate sines corresponding to any division of the quadrant, may be derived from the same principles: For the successive differences of the sines of the arcs A—B, A, and A+B, are $\sin A - \sin(A-B)$, and $\sin(A+B) - \sin A$; and consequently the difference between these again, or the *second difference* of the sines, is $\sin(A+B) + \sin(A-B) - 2\sin A =$ (Prop. 3. cor. 3. Trig.) $-2\,\mathrm{vers}B\,\sin A$. The second differences of the progressive sines are hence subtractive, and always proportional to the sines themselves. Wherefore the sines may be deduced from their second differences, by reversing the usual process, and recompounding their separate elements. Thus, the sines of A—B and A being already known, their second and descending difference, as it is thus derived from the sine of A, will combine to form the succeeding sine of A+B, which is $-2\,\mathrm{vers}B\sin A + (\sin A - \sin(A-B)) + \sin A$. It only remains then, to determine, in any trigonometrical system, the constant multiplier of the sine, or twice the versed sine of the component arc. Suppose the quadrant to be divided into 24 equal parts, each containing 3° 45′, or 225′. The length of this arc is nearly $\frac{22}{7}.\frac{1}{48} \doteq \frac{11}{168}$, and consequently twice its versed sine $= (\frac{11}{168})^2 =$ $(\frac{1}{233})$ in approximate terms. If the successive sines, corresponding to the division of the quadrant into 24 equal parts, be therefore continually multiplied by the fraction $\frac{1}{233}$, or divided by the number 233, the quotients thence arising will represent their second differences. But, since 233 is nearly equal to 225, or the length in minutes of the primary or component arc, and which differs not sensibly from its sine,—this last may be assumed as the divisor, the small aberration so produced being corrected by deferring the integral quotients. In this way the following Table is constructed:

Parts of the quadrant.	Arcs.	Sines.	1st Diff.	2d Diff.	Arcs.
1	225′	225	224	1	3° 45′
2	450′	449	222	2	7° 30′
3	675′	671	219	3	11° 15′
4	900′	890	215	4	15° 0′
5	1125′	1105	210	5	18° 45′
6	1350′	1315	205	* 5	22° 30′
7	1575′	1520	199	6	26° 15′
8	1800′	1719	191	* 7	30° 0′
9	2025′	1910	183	8	33° 45′
10	2250′	2093	174	9	37° 30′
11	2475′	2267	164	10	41° 15′
12	2700′	2431	154	* 10	45° 0′
13	2925′	2585	143	11	48° 45′
14	3150′	2728	131	12	52° 30′
15	3375′	2859	119	* 12	56° 15′
16	3600′	2978	106	13	60° 0′
17	3825′	3084	93	13	63° 45′
18	4050′	3177	79	14	67° 30′
19	4275′	3256	65	14	71° 15′
20	4500′	3321	51	14	75° 0′
21	4725′	3372	37	* 14	78° 45′
22	4950′	3409	22	15	82° 30′
23	5175′	3431	7	15	86° 15′
24	5400′	3438	0	15	90° 0′

The number 225, which expresses the length of the component arc, and consequently represents very nearly its sine, is here employed as the constant divisor. Thus, 225, divided by 225, gives a quotient 1, and this, subtracted from 225, leaves 224, which, being joined to 225, forms 449, the sine of the second arc. Again, 449 divided by 225, gives 2 for its integral quotient, which taken from 224, leaves 222; and this, added to 449, makes 671, the sine of the third arc. In this way, the sines are successively formed, till the quadrant is completed. The integral quotients, however, are deferred; that is, the nearest whole number in advance is not always taken: Thus the quotient of 1315 by 225, is $5\frac{38}{45}$, which approaches nearer to 6, and yet 5 is still retained. These efforts to redress the errors of computation are marked with asterisks.

It should be observed, that each of the three composite columns really forms a recurring series, In the second quadrant, the first dif-

ferences become subtractive, and the same numbers for the sines are repeated in an inverted order. By continuing the process, these sines are reproduced in the third and fourth quadrants, only on the opposite side.

Such is the detailed explication of that very ingenious mode, which, in certain cases, the Hindu astronomers employ, for constructing the table of approximate sines. But, ignorant totally of the principles of the operation, those humble calculators are content to follow blindly a slavish routine. The Brahmins must, therefore, have derived such information from people farther advanced than themselves in science, and of a bolder and more inventive genius. Whatever may be the pretensions of that passive race, their knowledge of trigonometrical computation has no solid claim to any high antiquity. It was probably, before the revival of letters in Europe, carried to the East, by the tide of victory. The natives of Hindustan might receive instruction from the Persian astronomers, who were themselves taught by the Greeks of Constantinople, and stimulated to those scientific pursuits by the skill and liberality of their Arabian conquerors.

The same principles lead to an elegant construction of the approximate sines, entirely adapted to the decimal scale of numeration, and the nautical division of the circle. Suppose a quadrant to contain 16 equal parts, or *half points*; the length of each arc is nearly $\frac{22}{7}.\frac{1}{32}=\frac{11}{112}$, and consequently twice its versed sine is $(\frac{11}{112})^2$, or, in round numbers, $\frac{1}{103}$. It will be sufficiently accurate, therefore, to employ 100 for the constant divisor. The sine of the first being likewise expressed by 100, let the nearer integral quotients be always retained, and the sine of the whole quadrant, or the radius itself, will come out exactly 1000. The first term being divided by 100 gives 1 for the second difference, which, subtracted from 100, leaves 99 for the first difference, and this joined to 100, forms the second term. Again, dividing 199 by 100, the quotient 2 is the second difference, which, taken from 99, leaves 97 for the first difference, and this added to 199, gives the third term. In like manner, the rest of the terms are found.

Half points.	Arcs.	Sines.	1st Diff.	2d Diff.	Excess.	Correct Sines.
1	5° 37½′	100	99	1	3	97
2	11° 15′	199	97	2	4	195
3	16° 52½′	296	94	3	5	291
4	22° 30′	390	90	4	6	384
5	28° 7½′	480	85	5	7	473
6	33° 45′	565	79	6	8	557
7	39° 22½′	644	73	6	9	635
8	45° 00′	717	66	7	10	707
9	50° 37½′	783	58	8	9	774
10	56° 15′	841	50	8	8	833
11	61° 52½′	891	41	9	7	884
12	67° 30′	932	32	9	6	926
13	73° 7½′	964	22	10	5	959
14	78° 45′	986	12	10	4	982
15	84° 22½′	998	2		3	995
16	90° 00′	1000				

The errors occasioned by neglecting the fractions accumulate at first, but afterwards gradually diminish, from the effect of compensation. The greatest deviation takes place, as might be expected, at the middle arc, whose sine is 707 instead of 717. Reckoning the error in excess as limited by 10, and declining uniformly on each side, the correct sines are finally deduced. The numbers thus obtained seldom differ, by the thousandth part, from the truth, and are hence far more accurate than the practice of navigation ever requires. This simple and expeditious mode of forming the sines is not merely an object of curiosity, but may be deemed of very considerable importance, as it will enable the mariner, altogether independent of the aid of books, to the loss of which he is often exposed by the hazards of the sea, to construct a table of *departure* and *difference of latitude*, sufficiently accurate for every real purpose.

Note LXXIII.—Page 367.

In trigonometrical investigations, it is often requisite to determine the proportion which the *difference* of an arc bears to that of its related lines. With this view, let Δ denote the increment or finite difference of the quantity to which it is prefixed.

1. In art. 29. of Note LXVI. change A into $A+\Delta A$, and B into A; then will

$\Delta sin A = 2 sin \frac{1}{2}\Delta A cos(A + \frac{1}{2}\Delta A)$.

2. Make the same change in art. 31. of that Note, and

$\Delta cos A = -2 sin \frac{1}{2}\Delta A sin(A + \frac{1}{2}\Delta A)$.

3. In art. 2. of Note LXX. let a similar change be made, and

$$\Delta tan A = \frac{sin \Delta A}{cos A cos(A + \Delta A)}.$$

4. Do the same thing in art. 4. and

$$\Delta cot A = -\frac{sin \Delta A}{sin A sin(A + \Delta A)}.$$

5. In art. 22. of Note LXVI. make a like substitution, and

$\Delta sin A^2 = sin \Delta A sin(2A + \Delta A)$.

6. Let the same change be made in art. 23., and

$\Delta cos A^2 = -sin \Delta A sin(2A + \Delta A)$.

7. Do the same thing in art. 16. of Note LXX. and

$$\Delta tan A^2 = \frac{sin \Delta A(sin 2A + \Delta A)}{cos A^2 cos(A + \Delta A)^2}.$$

8. Lastly, let a similar change be made in art. 17. of that Note, and

$$\Delta cot A^2 = -\frac{sin \Delta A(sin 2A + \Delta A)}{sin A^2 sin(A + \Delta A)^2}.$$

If the *differences* be conceived to diminish indefinitely and pass into *differentials*, these expressions, in coming to denote only limiting ratios, will drop their excrescences and acquire a much simpler form. Thus, adopting the characteristic *d*, since the ratio of an arc to its sine is ultimately that of equality, and the sine of $A + dA$ may be considered as the same with the sine of A; it follows, that

1. $d sin A = + cos A dA$.
2. $d cos A = - sin A dA$.
3. $d tan A = + \frac{dA}{cos A^2}$.
4. $d cot A = - \frac{dA}{sin A^2}$.
5. $d sin A^2 = + 2 sin A cos A dA$.
6. $d cos A^2 = - 2 sin A cos A dA$.
7. $d tan A^2 = + \frac{2 tan A dA}{cos A^2}$.
8. $d cot A^2 = - \frac{2 cot A dA}{sin A^2}$.

Note LXXIV.—Page 379.

Since, by Note LXXIII, $d\,sinA = cosA\,dA$, or the variation of the sine of an arc is proportional to its cosine; it follows that, near the termination of the quadrant, the slightest alteration in the value of a sine would occasion a material change in the arc itself. Again, from the same Note, $d\,tanA = \frac{dA}{cosA^2}$, or the variation of the tangent is inversely as the square of the cosine, and must therefore increase with extreme rapidity as the arc approaches to a quadrant.

Note LXXV.—Page 379.

It is convenient to reduce the solution of triangles to algebraic *formulæ*. Let a, b and c denote the sides of any plane triangle, and A, B, and C their opposite angles. The various relations which connect these quantities may all be derived from the application of Prop. 11.

1. $Cos\ A = \frac{b^2+c^2-a^2}{2bc}$.

2. But, since (art. 16. Note LXVI.) $sin\frac{1}{2}A^2 = \frac{1}{2}(1-cosA)$, it follows, by substitution, that $sin\frac{1}{2}A^2 = \frac{2bc-b^2-c^2+a^2}{4bc} = \frac{a^2-(b-c)^2}{4bc} = \frac{(a+b-c)(a-b+c)}{4bc}$, and therefore, s denoting the semiperimeter, $Sin\frac{1}{2}A^2 = \frac{(s-b)(s-c)}{bc}$; which corresponds to Prop 14.

3. Again, because (art. 17. Note LXVI.) $cos\frac{1}{2}A^2 = \frac{1}{2}(1+cosA)$, by substitution, $cos\frac{1}{2}A^2 = \frac{2bc+b^2+c^2-a^2}{4bc} = \frac{(b+c)^2-a^2}{4bc} = \frac{((b+c)+a)((b+c)-a)}{4bc}$, and consequently $Cos\frac{1}{2}A^2 = \frac{s(s-a)}{bc}$; which agrees with Prop. 13.

4. The second expression being now divided by the third, gives $Tan\frac{1}{2}A^2 = \frac{(s-b)(s-c)}{s(s-a)}$, corresponding to Prop. 12.

These are the *formulæ* wanted for the solution of the first case of oblique angled triangles. To obtain the rest, another transformation is required.

5. It is manifest that $sinA^2=1-cosA^2=\frac{4b^2c^2-(b^2+c^2-a^2)^2}{4b^2c^2}$, and consequently, by Note XLIX., $sinA^2=\frac{4T^2}{b^2c^2}$, or $sinA=\frac{2T}{bc}$. For the same reason, $sinB=\frac{2T}{ac}$, and hence $\frac{sinA}{sinB}=\frac{a}{b}$; which corresponds to Prop. 9.

6. Again, by composition, $\frac{sinA-sinB}{sinA+sinB}=\frac{a-b}{a+b}$, and therefore, by art. 18. Note LXX.,
$\frac{a-b}{a+b}=\frac{tan\frac{1}{2}(A-B)}{tan\frac{1}{2}(A+B)}$, which agrees with Prop. 10.

7. Lastly, transforming the first expression, there results,
$a=\sqrt{(b^2+c^2-2bc\,cosA)}=\sqrt{((b-c)^2+2bc\,versA)}$
$=\sqrt{((b+c)^2-2bc(1+cosA))}$.

The preceding *formulæ* will solve all the cases in plane trigonometry; but, by certain modifications, they may be sometimes better adapted for logarithmic calculation.

8. Divide the terms of art. 6. by a, and $\frac{tan\frac{1}{2}(A-B)}{tan\frac{1}{2}(A+B)}=\frac{1-\frac{b}{a}}{1+\frac{b}{a}}$; let $\frac{b}{a}=tan\ x$, and $\frac{tan\frac{1}{2}(A-B)}{tan\frac{1}{2}(A+B)}=\frac{1-tan\ x}{1+tan\ x}=$ (art. 6. Note LXX.) $tan\ (45^\circ-x)$. Wherefore $\frac{b}{a}=tan\ x$, and $tan(45^\circ-x)=$ $tan\frac{1}{2}C\ tan\frac{1}{2}(A-B)=tan\frac{1}{2}C\ cot(\frac{1}{2}C+B)=$ $tan\frac{1}{2}C(-cot(\frac{1}{2}C+A))$.

9. Again, from art. 7. $a=\sqrt{((b-c)^2+2bc\ versA)}=$ $(b-c)\sqrt{(1+\frac{2bc}{(b-c)^2}.\ versA)}$; consequently find $tan\ x=$ $\frac{\sqrt{2bc}}{b-c}\sqrt{vers\ A}=2\frac{\sqrt{bc}}{b-c}sin\frac{1}{2}A$, and $a=(b-c)\ sec\ x=\frac{b-c}{cos\ x}$.

10. But the expression in art. 1., by a different decomposition,

gives $a=\sqrt{((b+c)^2-2bc\ suvers A)}=(b+c)\sqrt{(1-\frac{2bc}{(b+c)^2}suvers A)}$; wherefore find $\sin x=\frac{\sqrt{2bc}}{b+c}\sqrt{suvers A}=2\frac{\sqrt{bc}}{b+c}\cos\frac{1}{2}A$, and $a=(b+c)\cos x$.

11. Other expressions are likewise occasionally used. Thus, by art. 1, $2bc.\cos A=b^2+c^2-a^2$, or $c^2-2bc.\cos A=a^2-b^2$, and, solving this quadratic, we obtain $c=b\cos A\pm\sqrt{(a^2-b^2+b^2\cos A^2)}=b\cos A\pm\sqrt{(a^2-b^2\sin A^2)}$, or $c=b\cos A\pm\sqrt{((a+b\sin A)(a-b\sin A))}$. When two sides and an angle opposite to one of them are given, the third side is thus found by a direct process.

12. From art. 5, $c=a\frac{\sin C}{\sin A}$; but C being a supplementary angle, its sine is the same as that of A + B, and consequently $c=a(\frac{\sin A\cos B+\cos A\sin B}{\sin A})$. By a similar transformation,

$$c=a\frac{\sin C}{\sin(B+C)}=a(\frac{\sin C}{\sin B\cos C+\cos B\sin C})=\frac{a}{\cos B+\sin B\cot C}.$$

13. Lastly, from art. 3. of Note LXX, $\cot A+\cot C=\frac{\sin(A+C)}{\sin A\sin C}$

$$=\frac{\sin B}{\sin A\sin C}=\frac{b}{a\sin C},$$ and therefore $\cot A=\frac{b}{a\sin C}-\cot C=\frac{b-a\cos C}{a\sin C}$, or $\tan A=\frac{a\sin C}{b-a\cos C}$.

If the angle A be assumed equal to 90°, the preceding *formulæ* will become restricted to the solution of right-angled triangles.

14. From art. 1, $\cos A=o=\frac{b^2+c^2-a^2}{2bc}$; whence, $a^2=b^2+c^2$, which expresses the radical property of the right angled triangle.

15. From art. 5, $\frac{\sin B}{\sin A}=\frac{b}{a}$, and consequently $\sin B=\frac{b}{a}$, which corresponds with Prop. 7.

16. Again, from the same article, $\frac{b}{c}=\frac{\sin B}{\sin C}=\frac{\sin B}{\cos B}$, and therefore $\tan B=\frac{b}{c}=\cot C$.

For the convenience of computing with logarithms, other expressions may be produced.

17. Thus, from art. 14., $b^2=a^2-c^2$, and hence $b=\sqrt{((a+c)(a-c))}$.

18. Since $a^2=b^2(1+\frac{c^2}{b^2})$, put $\frac{c}{b}=\tan x$, and $a=b(\sec x)=\frac{b}{\cos x}$.

19. Lastly, because $b^2=a^2(1-\frac{c^2}{a^2})$, put $\frac{c}{a}=\sin x$, and $b=a.\cos x$.

Besides the regular cases in the solution of triangles, other combinations of a more intricate kind sometimes occur in practice. It will suffice here to notice the most remarkable of these *varieties*.

20. Thus, suppose a side, with its opposite angle and the sum or difference of the containing sides, were given, to determine the triangle. By art. 5., $a=\frac{b \sin A}{\sin B}=\frac{c \sin A}{\sin C}$, and therefore

$a=\frac{b \sin A+c \sin A}{\sin B+\sin C}=\frac{(b+c)\sin(B+C)}{\sin B+\sin C}=$ (art. 5. and 18. Note LXVI.) $\frac{(b+c)2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B+C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}=\frac{(b+c)\cos\frac{1}{2}(B+C)}{\cos\frac{1}{2}(B-C)}$.

But $\cos\frac{1}{2}(B+C)=\sin\frac{1}{2}A$, and hence $\cos\frac{1}{2}(B-C)=\frac{(b+c)\sin\frac{1}{2}A}{a}$; and the difference of the supplementary angles B and C being known, these angles themselves are hence found.

In like manner, it will be found that $\sin\frac{1}{2}(B-C)=\frac{(b-c)\cos\frac{1}{2}A}{a}$.

21. Let a side with its adjacent angle and the sum of the other sides be given, to determine the triangle. By art. 4. $\tan\frac{1}{2}A^2=\frac{s-b.s-c}{s.s-a}$ and $\tan\frac{1}{2}B^2=\frac{s-a.s-c}{s.s-b}$; whence $\tan\frac{1}{2}A^2 \tan\frac{1}{2}B^2=\frac{s-a.s-b.(s-c)^2}{s-a.s-b.s^2}$, and consequently $\tan\frac{1}{2}A\tan\frac{1}{2}B=\frac{s-c}{s}=\frac{(a+b)-c}{(a+b)+c}$, or $\cot\frac{1}{2}B=\tan\frac{1}{2}A\frac{(a+b)+c}{(a+b)-c}$.

Again by art. 1, $2bc \cos A=b^2+c^2-a^2$, or $a^2-b^2-c^2=-2bc.\cos A$, and adding $2ab+2b^2$ to both sides, $a^2+2ab+b^2-c^2=2ab+2b^2-2bc.\cos A$, or $(a+b)^2-c^2=2b(a+b-c.\cos A)$; whence $((a+b)+c)$

$((a+b)-c)=2b(a+b-c.\cos A)$, and $b=\frac{1}{2}\frac{((a+b)+c)((a+b)-c)}{(a+b)-c.\cos A}$,

If the sign of b be changed, and the supplement of its adjacent angle therefore assumed, we shall obtain

$\cot\frac{1}{2}B=\tan\frac{1}{2}A\frac{c+(a-b)}{c-(a-b)}$, and $b=\frac{1}{2}\frac{((c-(a-b))(c+(a-b))}{c.\cos A-(a-b)}$.

The relation of the sides and angles of a triangle might also be in some cases conveniently expressed by a converging series. Thus $\frac{b}{a}=\frac{\sin B}{\sin A}=\frac{\sin B}{\sin(B+C)}=\frac{\sin B}{\sin B\cos C+\cos B\sin C}$, and consequently $b\sin B\cos C+b\cos B\sin C=a\sin B$, or $\frac{b\sin C}{a-b\cos C}=\frac{\sin B}{\cos B}=\tan B$.

Wherefore, by actual division, $\tan B=\frac{b}{a}\sin C+\frac{b^2}{a^2}\sin C\cos C+\frac{b^3}{a^3}\sin C\cos C^2+\frac{b^4}{a^4}\sin C\cos C^3+$ &c.; and, substituting the powers of this expression for those of the tangent in the series of Note LXXI., we obtain $B=\frac{b}{a}\sin C+\frac{b^2}{a^2}\sin C\cos C+\frac{b^3}{3a^3}(4\cos C^2-1)\sin C+\frac{b^4}{a^4}(2\cos C^2-1)\sin C\cos C+$ &c.; or $\frac{b}{a}\sin C+\frac{b^2}{2a^2}\sin 2C+\frac{b^3}{3a^3}\sin 3C+\frac{b^4}{4a^4}\sin 4C+$ &c.

In certain extreme cases, approximations can likewise be employed with advantage. Thus, suppose the angles A and B to be exceedingly small; then, by the last paragraph of page 364, their versed sines are very nearly equal to half the squares of the sines. Wherefore, $\sin C$, or $\sin(A+B)=$ (art. 1. Note LXVI.), $\sin A(1-\frac{1}{2}\sin B^2)+\sin B(1-\frac{1}{2}\sin A^2)$ nearly, and consequently, by art. 5., $c=(a+b)(1-\frac{1}{2}\sin A\sin B)$; or, the arcs being nearly equal to their sines, substitute c for $a+b$ in the second or differential term, and $c=a+b-\frac{1}{2}cAB$. Again, put $C=\pi-\theta$, or $\theta=A+B$, and $(a+b)(\frac{1}{2}\sin A\sin B)=\frac{1}{2}\sin A\sin B\frac{(a+b)^2}{a+b}=\frac{1}{2}\frac{ab\theta^2}{a+b}$ nearly, or $c=a+b-\frac{1}{2}\frac{ab\theta^2}{a+b}$.

Note LXXVI.—Page 388.

This problem, which is employed with great advantage in maritime surveying, admits likewise of a convenient analytical solution. Let the given distances AB, BC and AC be denoted by a, b and c, and the observed angles ADB and CDB by m and n; then (art. 5. Note LXXV.) $BD=\frac{a\ sin BAD}{sin\ m}=\frac{b\ sin BCD}{sin\ n}$, or $\frac{b\ sin\ m}{a\ sin\ n}=\frac{sin BAD}{sin BCD}$ and $\frac{b\ sin\ m-a\ sin\ n}{b\ sin\ m+a\ sin\ n}=\frac{sin BAD-sin BCD}{sin BAD+sin BCD}=$ (art. 18. Note LXX.) $\frac{tan\frac{1}{2}(BAD-BCD)}{tan\frac{1}{2}(BAD+BCD)}$. But the angles ABC and ADC of the quadrilateral figure DABC being evidently given, the sum of the remaining angles BAD and BCD is given, and each of them is consequently found. Hence the triangles ABD and CBD are immediately determined.

This most useful problem was first proposed by Mr Townley, and solved, in its various cases, by Mr John Collins, in the Philosophical Transactions for the year 1671. The second solution given in the text is borrowed from Legendre.

Note LXXVII.—Page 390.

This useful problem is commonly solved by the help of spherical trigonometry. It admits, however, of a simple and elegant general solution, derived from the arithmetic of sines. Let a and b denote the two vertical angles, or the acclivities of the diverging lines, A the oblique angle which these contain, and A′ the reduced or horizontal angle. Since the magnitude of an angle depends not on the length of its sides, assume each of them equal to the radius or unit, and it is evident that the base of the isosceles triangle thus limited will be the chord of the oblique angle A, the perpendiculars from its extremities to the horizontal plane, the

sines,—and the horizontal traces or projections, the cosines, of the vertical angles a and b. The base of the isosceles triangle forms the hypotenuse of a right-angled vertical triangle, of which the perpendicular is the difference between the vertical lines. Consequently the square of the reduced base is equal to the excess of the square of the chord of A above the square of the difference of the sines of a and b, or

(cor. 6. def. Trig.) $2-2\cos A-(\sin a-\sin b)^2=$
(II. 18. El.) $2-2\cos A-\sin a^2-\sin b^2+2\sin a\sin b=$
(2. cor. def. Trig.) $\cos a^2+\cos b^2+2\sin a\sin b-2\cos A$.

Wherefore (Prop. 11. Trig.) in the triangle now traced on the horizontal plane, $2\cos a\cos b\cos A'=2\cos A-2\sin a\sin b$; and multiplying by $\frac{1}{2}\sec a\sec b$, there results (cor. 4. def. Trig.),

1. $\cos A'=\sec a\sec b\cos A-\tan a\tan b$.

This expression appears concise and commodious, but it may be still variously transformed.

For $\text{vers}\,A'=1-\cos A'=1+\tan a\tan b-\sec a\sec b\cos A$ $=\sec a\sec b(\cos a\cos b+\sin a\sin b-\cos A)=$
(Prop. 2. Trig.) $\sec a\sec b(\cos(a-b)-\cos A)$; whence

2. $\text{Vers}\,A'=\sec a\sec b(\text{vers}\,A-\text{vers}(a-b).)$

Again, because (2. cor. 1. and 3. cor. 5. Trig.) $\text{vers}\,A'=2\sin\frac{1}{2}A'^2$, and $\text{vers}\,A-\text{vers}(a-b)=2\sin\frac{A+(a-b)}{2}\cdot\sin\frac{A-(a-b)}{2}$, we obtain, by substitution,

3. $\sin\frac{1}{2}A'^2=\sec a\sec b(\sin\frac{A+(a-b)}{2}.\sin\frac{A-(a+b)}{2})$.

Of these *formulæ*, the first, I presume, is new, and appears distinguished by its simplicity and elegance. The last one however is, on the whole, the best adapted for logarithmic calculation.

When the vertical angles are small, the problem will admit of a very convenient approximation. For, assuming the arcs a, b as equal to their tangents, it follows, by substitution, that $\cos A'=$ $\cos A\sqrt{(1+a^2)}\sqrt{(1+b^2)}-ab=\cos A((1+\frac{1}{2}a^2)(1+\frac{1}{2}b^2))-ab$ $=\cos A(1+\frac{1}{2}a^2+\frac{1}{2}b^2)-ab$, nearly. Whence, by Note LXXIII., the decrement of the cosine of that oblique angle is

$$ab - \cos A(\tfrac{1}{2}a^2 + \tfrac{1}{2}b^2);\ \text{but}$$

$$(\text{II. 19. El.})\ ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2,\ \text{and}$$

$$(\text{II. 21. El.})\ \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2 = \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2;$$

wherefore the decrement of $\cos A' =$

$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 - \cos A\left(\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2\right) =$$

$$\left(\frac{a+b}{2}\right)^2(1 - \cos A) - \left(\frac{a-b}{2}\right)^2(1 + \cos A).$$

Consequently the increment of the oblique angle itself is, by Note LXXIII,

$$\left(\frac{a+b}{2}\right)^2\left(\frac{1-\cos A}{\sin A}\right) - \left(\frac{a-b}{2}\right)^2\left(\frac{1+\cos A}{\sin A}\right) = \text{(art. 15. Note LXX.)}$$

$$\left(\frac{a+b}{2}\right)^2 \tan\tfrac{1}{2}A - \left(\frac{a-b}{2}\right)^2 \cot\tfrac{1}{2}A.$$

Such is the theorem which the celebrated Legendre has given, for reducing an oblique angle to its projection on the horizontal plane. It is very neat, and extremely useful in practice. But to connect it with our division of the quadrant, requires some adaptation. Let a and b express the vertical angles in minutes; then will $\left(\left(\frac{a+b}{2}\right)^2 \tan\tfrac{1}{2}A - \left(\frac{a-b}{2}\right)^2 \cot\tfrac{1}{2}A\right)\frac{1}{3438}$ denote, likewise in minutes, the quantity of reduction to be applied to the oblique angle.

In computing very extensive surveys, it becomes necessary to allow for the minute derangements occasioned by the convexity of the surface of our globe. The sides of the triangles which connect the successive stations, though reduced to the same horizontal plane, may be considered as formed by arcs of great circles, and their solution hence belongs to Spherical Trigonometry. But, avoiding such laborious calculations, for which indeed our Tables are not fitted, it seems far better to estimate merely the deviation of those incurved triangles from triangles with rectilineal sides. For effecting that correction two ingenious methods have lately been proposed on the Continent.

The first is that employed by M. Delambre, who substitutes the chords for their arcs, and thus converts the small spherical, into a plane, triangle. This conversion requires two distinct steps. 1. Each spherical angle, or that formed by tangents at the surface of the globe, is changed into its corresponding plane angle contained by the chords. Let α and β express the sides or arcs in miles; and the angles of elevation, or those made by the tangents and the respective chords, will be (III. 29. El.) denoted by $\frac{21600}{24856}\frac{1}{2}\alpha$ and $\frac{21600}{24856}\frac{1}{2}\beta$ in minutes, or $\frac{1350'}{3107}\alpha$ and $\frac{1350'}{3107}\beta$. Insert these values, therefore, in place of a and b in the *formula* of the preceding note, and the quantity of reduction of the angle A, contained by the small arcs α and β, will be $\left((\alpha+\beta)^2 \tan\frac{1}{2}A-(\alpha-\beta)^2 \cot\frac{1}{2}A\right)\frac{1''}{1214}$ in seconds. 2. Each arc is converted into its chord: But, by the Scholium to Proposition VI. of the Trigonometry, an arc α is to its chord, as 1 to $1-\frac{\alpha^2}{6D^2}$; wherefore the diminution of that arc in passing into its chord, amounts to the $\frac{\alpha^2}{375,600,000}$ part of the whole.

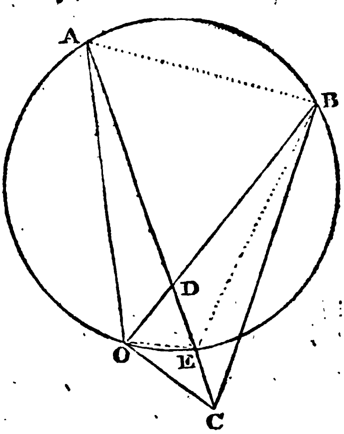
These reductions bestow great accuracy, and are sufficiently commodious in practice. But the second method of correcting the effects of the earth's convexity, and which was given by M. Legendre, is distinguished by its conciseness and peculiar elegance. That profound geometer viewed the spherical triangle as having its curved sides stretched out on a plane, and sought to determine the variation which its angles would thence undergo. Analysis led him, through a complicated process, to the discovery of a theorem of singular beauty. But the following investigation, grounded on other principles, appears to be much simpler.

Let A and B denote any two angles in the small spherical triangle, and α and β express in miles the opposite sides, or those of its extension upon a plane. Since (Prop. 9. Trig.) $\alpha : \beta :: \sin A : \sin B$, there must exist some minute arc θ, such that $\sin\alpha : \sin\beta :: \sin(A+\theta) : \sin(B+\theta)$. But (art. 1. Note LXVI.) $\sin(A+\theta)=\sin A+\theta\cos A$, and

(Schol. Prop. VI. Trig.) $\sin\alpha = \alpha - \frac{\alpha^3}{6}$; whence $\alpha - \frac{\alpha^3}{6} : \beta - \frac{\beta^3}{6} ::$ $\sin A + \theta\cos A : \sin B + \theta\cos B$. Now $\beta : \alpha :: \sin B : \sin A$, and therefore, (V. 9. El.) $1 - \frac{\alpha^2}{6} : 1 - \frac{\beta^2}{6} :: \sin A \sin B + \theta \cos A \sin B :$ $\sin A \sin B + \theta \sin A \cos B$. But the first and second terms being very nearly equal, and likewise the third and fourth,—it is obvious that the analogy will not be disturbed, if each of those pairs be increased equally. Hence $1 : 1 + \frac{\alpha^2 - \beta^2}{6} :: \sin A \sin B : \sin A \sin B +$ $\theta(\sin A \cos B - \cos A \sin B)$; and since (Prop. I. Trig.) $\sin A \cos B$ $- \cos A \sin B = \sin(A - B)$, therefore (V. 10. El.) $1 : \frac{\alpha^2 - \beta^2}{6} ::$ $\sin A \sin B : \theta \sin(A - B)$. Consequently, since α and β are proportional to $\sin A$ and $\sin B$, $\theta(\sin A - B) = \sin A \sin B\left(\frac{\alpha^2 - \beta^2}{6}\right) =$ $\frac{\alpha\beta}{6}\left(\sin A^2 - \sin B^2\right) = $ (Proposition III. cor. 5. Trigonometry,) $\frac{\alpha\beta}{6}\left(\sin(A+B)\sin(A-B)\right)$, or $\theta = \frac{\alpha\beta}{6}\sin(A+B)$. But the sine of the sum of A and B is the same as that of their supplement C, or of the angle contained by the sides α and β; and consequently θ is the third part of $\frac{\alpha\beta}{2}\sin C$, the area of the triangle, *or the third part of the excess of the angles of the spherical, above those of the plane,* triangle. Wherefore the sines of the sides, which, in the spherical triangle, are as the sines of their opposite angles, are likewise proportioned, in the plane triangle, to the sines of those angles, increasing each by the common excess. It is hence evident, that the angles of the plane triangle are obtained from those of the spherical, by deducting from each the third part of the excess above two right angles, as indicated by the area of the triangle.

The whole surface of the globe being proportioned to 720°, that of a square mile will correspond to $\frac{720°}{24856 \times 7912}$, or the $\frac{1}{75.88}$ part of a second. Hence each angle of the small spherical triangle requires to be diminished by $\alpha\beta \sin C \frac{1}{455.28}$ in seconds.

Another problem of great use in the practice of delicate surveying is *to reduce angles to the centre of the station.* Instead of planting moveable signals at each point of observation, it will often be found more convenient to select the more notable spires, towers, or other prominent objects which occur interspersed over the face of the country. In such cases, it is evidently impossible for the theodolite or circular instrument, although brought within the cover of the building, to be placed immediately under the vane. The observer approaches the centre of the station as near, therefore, as he can with advantage, and calculates the quantity of error which the minute displacement may occasion. Thus, suppose it were required to determine the angle AOB which the remote objects A and B subtend at O, the centre of a permanent station: The instrument is placed in the immediate vicinity at the point C, and the distance CO, with the angle of deviation OCA, are noted, while the principal angle ACB is observed. The central angle AOB may hence be computed from the rules of trigonometry; but the calculation is effected by simpler and more expeditious methods. Since (I. 32. El.) the exterior angle ADB is equal both to AOB with OAC, and to ACB with OBC; it is evident that AOB = ACB + OBC — OAC. But the angles OBC and OAC, being extremely small, may be considered as equal to their sines, and (art. 5. Note LXXV.) $sin\, OBC = \frac{CO}{OB} sin\, BCO$, and



$sin\, OAC = \frac{CO}{OA} sin\, ACO$; wherefore the angle AOB at the centre, is nearly equal to $ACB + CO\left(\frac{sinBCO}{OB} - \frac{sinACO}{OA}\right) = ACB + CO\left(\frac{sin(ACB+ACO)}{OB} - \frac{sinACO}{OA}\right)$. Call the distances AC and BC of the point of observation, *a* and *b*, the distances AO and

BO of the centre a' and b'; the displacement CO, and the angle ACO of deviation m and ϕ, while the subtended angles ACB and AOB are denoted by C and C′, and the opposite angles ABO and OAB by A and B; then $C'=C+m\left(\frac{\sin(C+\phi)}{b'}-\frac{\sin\phi}{a'}\right)3438'$. If the centre O lies on AC, the correction of the observed angle, expressed in minutes, will be merely $\left(\frac{m}{b'}\sin C\right)3438$.

But the problem admits of a simpler approximation. Let a circle circumscribe the points A, O, and B, and cut AC in E. The angle AOB = (III. 18. El.) AEB = ACB + CBE; but $\sin CBE=\frac{CE}{EB}\sin ACB$, and $\sin OEC=\sin AEO$ or *ABO* is equal to $\frac{CO}{CE}\sin COE$ or AEO — ACO, and hence by combination $\sin CBE=\frac{CO}{EB}\frac{\sin ACB\sin(ABO-ACO)}{\sin ABO}$. Since, therefore, EB is nearly equal to OB, and the small angle CBE may be regarded as equal to its sine, the correction to be added to the observed angle is denoted in minutes by $\frac{m}{b'}\frac{\sin C\sin(A-\phi)}{\sin A}3438$. This quantity, it is evident, will entirely vanish when ϕ becomes equal to A, or the angle ABO equals ACO; in which case, the point of observation C coincides with E, or lies in the circumference of a circle that passes through the two remote points A and B and centre of the station. To place the instrument at E therefore, would only require to move it along CA, till the angle AEO be equal to ABO.

Both these methods for the reduction of an angle to the centre are given by Delambre; but, in his calculations, he generally preferred the last one, as being simpler and sufficiently accurate for practice. The investigation however will be found to be now considerably shortened.

The accuracy of trigonometrical operations must depend on the proper selection of the connecting triangles. It is very important, therefore, in practice, to estimate the variations which are produced among the several parts of a triangle, by any change of their mutual

relations. Suppose two of the three determining parts of a triangle to remain constant, while the rest undergo some partial change; and let, as before, the small letters *a*, *b* and *c* denote the sides of the triangle, and the capitals A, B and C their opposite angles.

Case I.—*When two sides a and b are constant.*

Since the angles A and B, after passing into $A+\Delta A$ and $B+\Delta B$, must have their sines still proportional to the opposite sides, it is evident that $\frac{\sin A}{\sin(A+\Delta A)}=\frac{\sin B}{\sin(B+\Delta B)}$, and consequently $\frac{\sin(A+\Delta A)-\sin A}{\sin(A+\Delta A)+\sin A}=\frac{\sin(B+\Delta B)-\sin B}{\sin(B+\Delta B)+\sin B}$; wherefore, by alternation and art. 7. Note LXX.,

1. $\frac{\tan\frac{1}{2}\Delta A}{\tan\frac{1}{2}\Delta B}=\frac{\tan(A+\frac{1}{2}\Delta A)}{\tan(B+\frac{1}{2}\Delta B)}$.

Next, in the incremental triangle formed by the sides c, $c+\Delta c$, and the contained angle ΔA, (art. 1. Note LXXV.) $\frac{\frac{1}{2}\Delta c}{c+\frac{1}{2}\Delta c}=-\frac{\tan(B+\frac{1}{2}\Delta B)}{\cot\frac{1}{2}\Delta A}$, and hence reciprocally,

2. $\frac{\frac{1}{2}\Delta c}{\tan\frac{1}{2}\Delta A}=-\frac{c+\frac{1}{2}\Delta c}{\cot(B+\frac{1}{2}\Delta B)}$.

In like manner, from the incremental triangle contained by the sides c, $c+\Delta c$ and the angle ΔB, it follows that

3. $\frac{\frac{1}{2}\Delta c}{\tan\frac{1}{2}\Delta B}=-\frac{c+\frac{1}{2}\Delta c}{\cot(A+\frac{1}{2}\Delta A)}$.

Again, the base of the incremental isosceles triangle contained by the equal sides b, b, and the vertical angle ΔC, is (art. 15. Note LXXV.) $2b\sin\Delta C$; wherefore, in the incremental triangle formed with the same base and the sides c and $c+\Delta c$, by art. 20. Note LXXV., $\cos(A+\frac{1}{2}\Delta A)=\frac{(c+\frac{1}{2}\Delta c)\sin\frac{1}{2}\Delta B}{b\sin\frac{1}{2}\Delta c}$; whence

4. $\frac{\sin\frac{1}{2}\Delta B}{\sin\frac{1}{2}\Delta C}=-\frac{b\cos(A+\frac{1}{2}\Delta A)}{c+\frac{1}{2}\Delta c}$.

After the same manner, it will be found that

5. $\frac{\sin\frac{1}{2}\Delta A}{\sin\frac{1}{2}\Delta C}=-\frac{a\cos(B+\frac{1}{2}\Delta B)}{c+\frac{1}{2}\Delta c}$.

Multiply the expressions of art. 4. into those of art 3. and

$$6.\ \frac{\frac{1}{2}\Delta}{\sin\frac{1}{2}\Delta C}=\frac{b\sin(A+\frac{1}{2}\Delta A)}{\cos\frac{1}{2}\Delta B}.$$

Multiply likewise the expressions of art. 2. and 5., and

$$7.\ \frac{\frac{1}{2}\Delta c}{\sin\frac{1}{2}\Delta C}=\frac{a\sin(B+\frac{1}{2}\Delta B)}{\cos\frac{1}{2}\Delta A}.$$

If, in all the preceding *formulæ*, the increments annexed to the varying quantities be omitted, there will arise much simpler expressions for the differentials.

$$1.\ \frac{dA}{dB}=\frac{\tan A}{\tan B}.$$

$$2.\ \frac{dc}{dA}=-\frac{c}{\cot B}.$$

$$3.\ \frac{dc}{dB}=-\frac{c}{\cot A}.$$

$$4.\ \frac{dB}{dC}=-\frac{b}{c}\cos A.$$

$$5.\ \frac{dA}{dC}=-\frac{a}{c}\cos B.$$

$$6.\ \frac{dc}{dC}=b\sin A.$$

$$7.\ \frac{dc}{dC}=a\sin B.$$

Case II.—*When one side a, and its opposite angle A, are constant.*

Since (art. 5. Note LXXV.) $\frac{a}{\sin A}=\frac{b}{\sin B}$, it is evident that $a\sin B=b\sin A$, and taking the differences by art. 1. of Note LXXIII. $\Delta b\sin A=2a\sin\frac{1}{2}\Delta B\cos(B+\frac{1}{2}\Delta B)$, whence $\frac{\sin\frac{1}{2}\Delta B}{\frac{1}{2}\Delta b}=\frac{\sin A}{a\cos(B+\frac{1}{2}\Delta B)}$, and consequently, by art. 5. of Note LXXV.

$$8.\ \frac{\sin\frac{1}{2}\Delta B}{\frac{1}{2}\Delta b}=-\frac{\sin\frac{1}{2}\Delta C}{\frac{1}{2}\Delta b}=\frac{\sin B}{b\cos(B+\frac{1}{2}\Delta B)}.$$

In like manner, it will be found that

$$9.\ \frac{\sin\frac{1}{2}\Delta B}{\frac{1}{2}\Delta c}=-\frac{\sin\frac{1}{2}\Delta C}{\frac{1}{2}\Delta c}=-\frac{\sin C}{c\cos(C+\frac{1}{2}\Delta C)}.$$

Combine the two last expressions, and

$$10.\ \frac{\Delta b}{\Delta c}=-\frac{\cos(B+\frac{1}{2}\Delta B)}{\cos(C+\frac{1}{2}\Delta C)}.$$

The differentials are discovered, by rejecting the modifications of the variable quantities.

$$* \ 8. \ \frac{dB}{db} = \frac{sinB}{b \ cosB} = \frac{tanB}{b}.$$

$$* \ 9. \ \frac{dB}{dc} = -\frac{sinC}{c \ cosC} = -\frac{tanC}{a}.$$

$$* \ 10. \ \frac{db}{dc} = -\frac{cosB}{cosC}.$$

Case III.—*When one side a, and its adjacent angle* B, *are constant.*

In the incremental triangle contained by the sides b, $b+\Delta b$, and Δc, it is evident, (art. 5. Note LXXV), that

$$11. \ \frac{\Delta c}{sin\Delta C} = -\frac{\Delta c}{sin\Delta A} = \frac{b}{sin(A+\Delta A)} = \frac{b+\Delta b}{sinA},$$

Again, in the same incremental triangle, (art. 6. Note LXXV.)

$$12. \ \frac{\frac{1}{2}\Delta b}{tan\frac{1}{2}\Delta C} = -\frac{\frac{1}{2}\Delta b}{tan\frac{1}{2}\Delta A} = \frac{b+\frac{1}{2}\Delta b}{tan(A+\frac{1}{2}\Delta A)}.$$

Or, transforming the preceding expression,

$\frac{\frac{1}{2}\Delta b}{b+\frac{1}{2}\Delta b} = -\frac{tan\frac{1}{2}\Delta A}{tan(A+\frac{1}{2}\Delta A)}$, and consequently

$$\frac{\frac{1}{2}\Delta b}{b} = -\frac{tan\frac{1}{2}\Delta A}{tan(A+\frac{1}{2}\Delta A)+tan\frac{1}{2}\Delta A} = \text{(art. 1. Note LXX.)}$$

$$-tan\tfrac{1}{2}\Delta A\left(\frac{cos(A+\frac{1}{2}\Delta A cos\frac{1}{2}\Delta A)}{sin(A+\Delta A)}\right) = -sin\tfrac{1}{2}\Delta A\left(\frac{cos(A+\frac{1}{2}\Delta A)}{sin(A+\Delta A)}\right);$$

wherefore,

$$13. \ \frac{\frac{1}{2}\Delta b}{sin\frac{1}{2}\Delta C} = -\frac{\frac{1}{2}\Delta b}{sin\frac{1}{2}\Delta A} = b\left(\frac{cos(A+\frac{1}{2}\Delta A)}{sin(A+\Delta A)}\right).$$

Again, in the same incremental triangle, by art. 20. Note LXXV.

$cos(A+\frac{1}{2}\Delta A) = \frac{\Delta b}{\Delta c}(-cos\frac{1}{2}\Delta C) = \frac{\Delta b}{\Delta c} cos\frac{1}{2}\Delta A$; whence

$$14. \ \frac{\Delta b}{\Delta c} = \frac{cos(A+\frac{1}{2}\Delta A)}{cos\frac{1}{2}\Delta A}.$$

The differentials are found as before, by the omission of the minute excrescences.

$$* \ 11. \ \frac{dc}{dC} = -\frac{dc}{dA} = \frac{b}{sinA}.$$

$$* \ 12. \ \frac{db}{dC} = -\frac{db}{dA} = \frac{b}{tanA}.$$

13. $\dfrac{db}{dC} = -\dfrac{db}{dA} = b\left(\dfrac{\cos A}{\sin A}\right) = b\cot A.$

14. $\dfrac{db}{dc} = \cos A.$

To compute the values of the finite differences, when these differences themselves are involved in their compound expression, the easiest method is to proceed by repeated approximations. Thus, from art. 3. $\Delta c = -\dfrac{\tan\frac{1}{2}\Delta B}{\cot(A+\frac{1}{2}\Delta A)}(2c+\Delta c)$; assume, therefore, first, $\Delta c = -\dfrac{\tan\frac{1}{2}\Delta B}{\cot(A+\frac{1}{2}\Delta A)}\,2c$; and then, $\Delta c = -\dfrac{\tan\frac{1}{2}\Delta B}{\cot(A+\frac{1}{2}\Delta A)}\left(2c - \dfrac{\tan\frac{1}{2}\Delta B}{\cot(A+\frac{1}{2}\Delta A)}\,2c\right)$. But it will seldom be requisite to advance beyond two steps; though the process, if continued, would evidently form an infinite converging series.

When only one part of a triangle remains constant, the expressions for the finite differences will often become extremely complicated. It may be sufficient in general to discover the relations of the differentials merely. To do this, let each indeterminate part be supposed to vary separately, and find, by the preceding *formulæ*, the effect produced; these distinct elements of variation being collected together, will exhibit the entire differential.

The materials of this intricate Note appear in Cagnoli, *but* the subject was first started by our countryman Mr Cotes, a mathematician of profound and original genius, in a brief tract, entitled *Estimatio errorum in mixtâ Mathesi*. It is unfortunate that I have not room for explaining the application of those *formulæ* to the selection and proper combination of triangles in nice surveys.

Having in some of the preceding notes briefly pointed out the several corrections employed in the more delicate geodesiacal operations, I shall subjoin a few general remarks on the application of trigonometry to practice. The art of surveying consists in determining the boundaries of an extended surface. When performed in the completest manner, it ascertains the positions of all the prominent objects within the scope of observation, measures their mutual distances and relative heights, and consequently defines the various

contours which mark the surface. But the land-surveyor seldom aims at such minute and scrupulous accuracy; his main object is to trace expeditiously the chief boundaries, and to compute the superficial contents of each field. In hilly grounds, however, it is not the absolute surface that is measured, but the diminished quantity which would result, had the whole been reduced to a horizontal plane. This distinction is founded on the obvious principle, that, since plants shoot up vertically, the vegetable produce of a swelling eminence can never exceed what would have grown from its levelled base. All the sloping or hypotenusal distances are, therefore, reduced invariably to their horizontal lengths, before the calculation is begun.

Land is surveyed either by means of the chain simply, or by combining it with a theodolite or some other angular instrument. The several fields are divided into large triangles, of which the sides are measured by the chain; and if the exterior boundary happens to be irregular, the perpendicular distance or offset is taken at each bending. The surface of the component triangles is then computed from Prop. 31. Book VI. of the Elements of Geometry, and that of the accrescent space by Note XV. to Prop. 10. Book II. In this method the triangles should be chosen as nearly equilateral as possible; for if they be very oblique, the smallest error in the length of their sides will occasion a wide difference in the estimate of the surface. The calculation is much simpler from the application of Prop. 6. Book II. of the Elements, the base and altitude of each triangle only being measured; but that slovenly practice appears liable to great inaccuracy. The perpendicular may indeed be traced by help of the surveying cross, or more correctly by the box sextant, or the optical square, which is only the same instrument in a reduced and limited form; yet such repeated and unavoidable interruption to the progress of the work will probably more than counterbalance any advantage that might thence be gained.

The usual mode of surveying a large estate, is to measure round it with the chain, and observe the angles at each turn by means of the theodolite. But these observations would require to be made with great care. If the boundaries of the estate be tolerably regular, it may be considered as a polygon, of which the angles, being

necessarily very oblique, are therefore apt to affect the accuracy of the results. It would serve to rectify the conclusions, were such angles at each station conveniently divided, and the more distant signals observed. The best method of surveying, if not always the most expeditious, undoubtedly is to cover the ground with a series of connected triangles, planting the theodolite at each angular point, and computing from some base of considerable extent, which has been selected and measured with nice attention. The labour of transporting the instrument might also in many cases be abridged, by observing at any station the bearings at once of several signals. Angles can be measured more accurately than lines, and it might therefore be desirable that surveyors would generally employ theodolites of a better construction, and trust less to the aid of the chain.

The quantity of surface marked out in this way, is easily computed from trigonometry. Adopting the general notation, the area of a triangle which has two sides, and their included angle known, it is evident, will be denoted by $\frac{ab}{2}.\sin C$, and the area of a triangle of which there are given all the angles and a side, is $\frac{a^2}{2}.\frac{\sin B \sin C}{\sin A}$.

The English *chain* is 22 yards, or 66 feet in length, and equivalent to four *poles*; it is hence the tenth part of a furlong, or the eightieth part of a mile. The chain is divided into a hundred links, each occupying 7.92 inches. An *acre* contains ten square chains or 100,000 links. A square mile, therefore, includes 640 acres; and this large measure is deemed sufficient, in certain rude and savage countries, as the Back Settlements of America, where vast tracts of new land are allotted merely by running lines north and south, and intersecting these by perpendiculars, at each interval of a mile.

The Scotch chain consists of 24 ells, each containing 37.069 inches, and ought therefore to have 74.138 feet for its correct length. The English acre is hence to the Scotch, in round numbers, as 11 to 14, or very nearly as the circle to its circumscribing square. But this provincial measure is gradually wearing into disuse, and already the statute acre seems to be generally adopted in the counties south of the Forth.

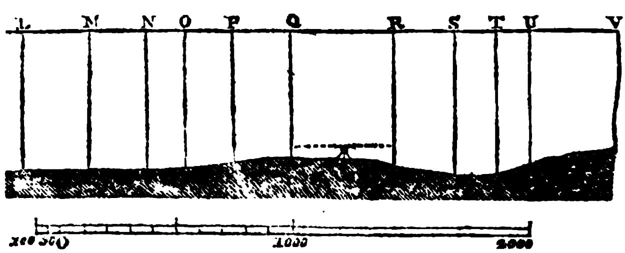
---

* LEVELLING is a delicate and important branch of general surveying. It may be performed very expeditiously by help of a large theodolite, capable of measuring with precision the vertical angle subtended by a remote object, the distance being calculated, and allowance made for the effect of the earth's convexity and the influence of refraction. But the more usual and preferable method is to employ an instrument designed for the purpose, and termed a *spirit-level*, which is accompanied by a pair of square staves, each composed of two parts that slide out into a rod of ten feet in length, every foot being divided centesimally. Levelling is distinguished into two kinds, the simple and the compound; the former, which rarely admits of application, assigns the difference of altitude by a single observation; but the latter discovers it from a combined series of observations carried along an irregular surface, the aggregate of the several descents being deducted from that of the ascents. The staves are therefore placed successively along the line of survey, at suitable intervals according to the nature of the ground and not exceeding 400 yards, the levelling instrument being always planted nearly in the middle between them, and directed backwards to the first staff, and then forwards to the second. The difference between the heights intercepted by the back and the fore observation, must evidently give at each station the quantity of ascent or descent, and the error occasioned by the curvature of the globe may be safely overlooked, as on such short distances it will not amount at each station to the hundredth part of a foot. To discover the final result of a series of operations, or the difference of altitude between the extreme stations, the measures of the back and fore observations are all collected severally, and the excess of the latter above the former indicates the entire quantity of descent.

As an example of levelling, I shall take the concluding part of a survey which my friend Mr Jardine, civil engineer, has recently made for the Town-Council of Edinburgh, with a degree of accuracy seldom attempted, in tracing the descent from the Black and Crawley springs, near the summits of the Pentland chain, to the Reservoir on the Castlehill, with a view to the conducting of a fresh supply of water from those heights, To avoid unnecessary complication,

however, I shall only notice the principal stations. The figure annexed represents a profile or vertical section of the ground, LV is the level of the Black spring, and the several perpendiculars from it denote the varying depth of the surface, referred to the base assumed 700 feet below. The stations marked are as follow:

L Lowest point in the Meadow.
M Cleansing cocks on the north side of the Meadow.
N Sunk fence in Lord Wemyss's garden,
O Air cock in Archibald's nursery.
P South side of Lauriston road
Q Bottom of Heriot's Green Reservoir.
R Head of Hamilton's close.
S Strand on south side of Grassmarket.
T Cleansing cock on north side of Grassmarket.
U Gaelic Chapel.
V Upper side of the belt of Castlehill Reservoir.



Stations.	Distance. Feet.	Back Observation. Feet.	Fore Observation. Feet.	Ascent. Feet.
L	—	—	—	—
M	370	4.59	2.04	2.55
N	640	8.68	3.05	8:18
O	905	9.12	2.22	15.08
P	1236	29.43	2.11	42.40
Q	1493	16.24	1.40	57.24
R	1925	2.54	26.98	32.80
S	2260	4.69	53.28	—15.79
T	2352	4.22	4.42	—15.99
U	2540	32.40	1.25	15.15
V	2705	94.77	9.97	99.95

Black spring, being 620.05 feet above the level of the Meadow, is therefore 520.1 feet higher than the belt of the reservoir. The numbers exhibited in the last column, are obtained by taking the differences of the aggregates of the two preceding columns. Where the ground either sinks or rises suddenly, some intermediate observations are here grouped together into a single amount. Thus, three observations were made between O and P, two between P and Q, three between Q and R, five between R and S, three between T and U, and no fewer than nine between U and V. The slight sketch between the perpendiculars from Q and R, shows the mode of planting and directing the instrument.

The mode of levelling on a grand scale, or determining the heights of distant mountains, will receive illustration from the third volume of the Trigonometrical Survey, which Colonel Mudge has been kindly pleased to communicate to me before its publication. I shall select the largest triangle in the series, being one that connects the North of England with the Borders of Scotland. The distance of the station on Cross Fell to that on Wisp Hill, is computed at 235018.6 feet, or 44.511 miles, which, reckoning 6094¾ feet for the length of a minute near that parallel, corresponds, on the surface of the globe, to an arc of 38′ 33″.7. Wisp Hill was seen depressed 30′ 48″ from Cross Fell; which again had a depression of 2′ 31″ when viewed from Wisp Hill. The sum of these depressions is 33′ 19″, which, taken from 38′ 33″.7, the measure of the intercepted arc, or the angle at the centre, leaves 5′ 14″.7, for the joint effect of refraction at both stations. The deflection of the visual ray produced by that cause, and which the French philosophers estimate in general at .079, had therefore amounted only to .06805, or a very little more than the *fifteenth* part of the intercepted arc. Hence, the true depression of Wisp Hill was 30′ 48″ — 16′ 39″.5 = 14′ 8″.5; and consequently, estimating from the given distance, it is 967 feet lower than Cross Fell.

From Wisp Hill, the top of Cheviot appeared exactly on the same level, at the distance of 185023.9 feet, or 35.0424 miles. Wherefore, two-thirds of the square of this last number, or 819, would, from the scholium at page 391. express in feet the approximate height of

Cheviot above Wisp Hill. But refraction gave the mountain a more towering elevation than it really had; and the measure being reduced in the former ratio of 38′ 33″.7 to 33′ 19″, is hence brought down to 708 feet.

Again, the distance 292012.7 feet, or 55.3054 miles, of Cross Fell from Cheviot, corresponds to an arc of 47′ 54″.8, which, reduced by the effect of refraction, would leave 41′ 23″.8 for the sum of the depressions at both stations. Consequently, Cheviot had, from Cross Fell, a true depression of only 23′ 44″—20′ 41″.9 or 3′ 2″.1, and is therefore lower than that mountain by 258 feet.

These results agree very nearly with each other. The height of Cross Fell above the level of the sea being 2901, that of *Wisp Hill* is 1934, and that of Cheviot 2642 or 2643. In the Trigonometrical Survey, the latter heights are stated at 1940 and 2658; a difference of small moment, owing to a balance of errors, or perhaps to the adoption of some other *data* with respect to horizontal refraction, and which do not appear on record.

From the same valuable work, I am tempted to borrow another example, which has more local interest. From Lumsdane Hill, the north top of Largo Law, at the distance of 189240.1 feet, or 35.84 miles, appeared sunk 9′ 32″ below the horizon. Here the intercepted arc is 31′ 3″ and the effect of the earth's curvature, modified by refraction, is 13′ 24″.8; whence the true elevation of Largo Law was 13′ 24″.8—9′ 32″, or 3′ 52″.8, which makes it 213 feet higher than Lumsdane Hill, or 938 feet above the level of the sea. In the Trigonometrical Survey, this height is stated at 952; but I am inclined to prefer the former number, having once found it by a barometrical measurement, in weather not indeed the most favourable, to be only 935 feet.

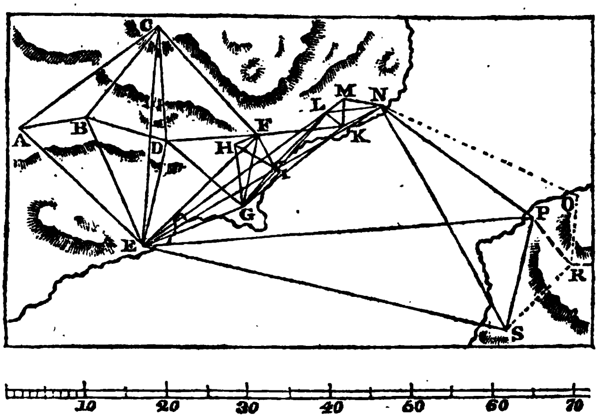
MARITIME SURVEYING is of a mixed nature: It not only determines the positions of the remarkable headlands, and other conspicuous objects that present themselves along the vicinity of a coast, but likewise ascertains the situation of the various inlets, rocks, shallows and soundings which occur in approaching the shore. To sur-

vey a new or inaccessible coast, two boats are moored at a proper interval, which is carefully measured on the surface of the water; and from each boat the bearings of all the prominent points of land are taken by means of an azimuth compass, or the angles subtended by these points and the other boat are measured by a Hadley's sextant. Having now on paper drawn the base to any scale, straight lines radiating from each end at the observed angles, as in Prop. 21. of the Trigonometry, will by their intersections give the positions of the several points from which the coast may be sketched.—But a chart is more accurately constructed, by combining a survey made on land, with observations taken on the water. A smooth level piece of ground is chosen, on which a base of considerable length is measured out, and station staves are fixed at its extremities. If no such place can be found, the mutual distance and position of two points conveniently situate for planting the staves, though divided by a broken surface, are determined from one or more triangles, which connect with a shorter and temporary base assumed near the beach. A boat then explores the offing, and at every rock, shallow, or remarkable sounding, the bearings of the station staves are noticed. These observations furnish so many triangles, from which the situation of the several points are easily ascertained.—When a correct map of the coast can be procured, the labour of executing a maritime survey is materially shortened. From each notable point of the surface of the water, the bearings of two known objects on the land are taken, or the intermediate angles subtended by three such objects are observed. In the first case, those various points have their situations ascertained by Prop. 21. and the second case by Prop. 25. of the Trigonometry. To facilitate the last construction, an instrument called the *Station-Pointer* has been invented, consisting of three brass rulers, which open and set at the given angles.

The nice art of observing has in its progress kept pace with the improved skill displayed in the construction of instruments. Surveys on a vast scale have lately been performed in Europe, with that refined accuracy which seems to mark the perfection of science. After the conclusion of the American war, a memoir of Count Cassini de Thury was transmitted by the French government to our

Court, stating the important advantages which would accrue to astronomy and navigation, if the difference between the meridians of the observations of Greenwich and Paris were ascertained by actual measurement. A spirit of accommodation and concert fortunately then prevailed. Orders were speedily given for carrying the plan into execution; and General Roy, who was charged with the conduct of the business on this side of the Channel, proceeded with activity and zeal. In the summer of 1784, a fundamental base, rather more than five miles in length, was traced on Hounslow Heath, about 54 feet above the level of the sea, and measured with every precaution, by means of deal rods, glass tubes, and a steel chain, allowance being made for the effects of the variable heat of the atmosphere in expanding those materials. The same line was, seven years afterwards, remeasured with an improved chain, which yet gave a difference on the whole of only three inches. The mean result, or 27404.2 feet, at the temperature of 62° by Fahrenheit's scale, is therefore assumed as the true length of the base. Connected with this line, and commencing from Windsor Castle, a series of thirty-two primary triangles was, in 1787 and 1788, extended to Dover and Hastings, on the coast of Kent and Sussex. Two triangles more stretched across the Channel. The horizontal and vertical angles at each station were taken with singular accuracy by a theodolite, which the celebrated artist Ramsden had, after much delay, constructed, of the largest dimensions and the most exquisite workmanship. At the same period, a new base of verification was measured on Romney Marsh, 15½ feet above the sea, and found, after various reductions, to be 28535.6773 feet in length. This base, computed from the nearest chain of triangles dependent on that of Hounslow Heath, ought to have been 28533.3; differing scarcely more than two feet on a distance of eighty miles. The mean, or 28534.5, is adopted for calculating the adjacent and subsequent triangles. These triangles near the coast were unavoidably confined and oblique; but their sides are generally deduced from larger and more regular triangles, expanding over the interior of the country. The annexed figure exhibits the most interesting portion of this memorable survey, and represents the various combination of triangles. Attached to it is a scale of English miles.

A Frant Church	K Folkstone Turnpike
B Goodhurst Church	L Padlesworth
C Hollingborn Hill	M Swingfield Church
D Tenterden Church	N Dover Castle
E Fairlight Down	O Church at Calais
F Allington Knoll	P Blancnez Signal
G Lydd Church	R Fiennes Signal
H Ruckinge	S Montlambert Signal
I High Nook	KL *The base of verification.*



Calculation of the sides of the Triangles.

			ACE	
A	70°	23′	2″	141744.4
C	52	11	3 *	113926
E	48	25	55	107895.7

			ABC	
A	27	4	36.13	71298.5
B	136	27	35.87	——
C	16	27	48 *	44391.2

			ABE	
A	43	18	25.87	93629.2
B	105	39	28.86	——
E	31	2	5.27	——

			BCD	
B	68	13	19.5	71887.5
C	44	38	44.04 *	54376.5
D	67	7	56.46	——

			BDE	
B	49°	39′	35.77″	71637.2
D	94	59	25.81	93629.2
E	35	20	58.42 *	——

			CDF	
C	40	0	58.96 *	61777.5
D	91	34	22.04	96039.8
F	48	24	39	——

			DFG	
D	43	45	23.18	47850.9
F	73	0	27	66169.2
G	63	14	9.82 *	——

			DEG	
D	62	32	52.51	71692.2
E	54	59	17.31	——
G	62	27	50.18 *	71637.2

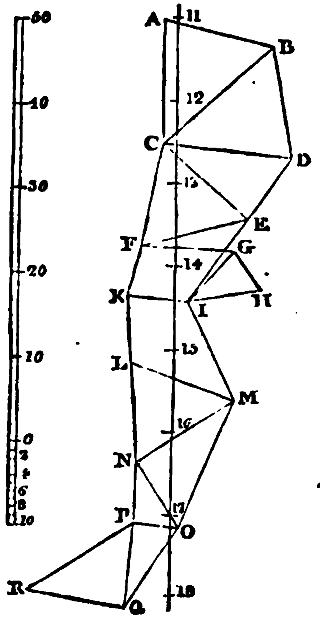
			EFG	
E	21°	18′	37″ *	47850.9
F	32	59	23	——
G	125	42	0	106926.2
			FGI	
F	33	8	46.1	31363.7
G	26	57	29.9 *	23185.7
I	·121	53	44	——
			FHI	
F.	91	27	19.5	28534.5
H	54	19	18.5	——
I	34	13	22	16053
			FGK	
F	109	50	39.35	84662.8
G	38	2	23.76	55463.6
K	32	6	56.89 *	——
			EGL	
E	13	38	2.95 *	79536.1
G	154	5	54.4	14739.2
L	12	16	2.65	——.
			FIK	
F	76	1	53.25	54708
I	79	41	0.5	——
K	24	17	6.25	——
			IKL	
I	14	48	25.5 *	14714.3
K	57	2	0	48305.2
L	108	9	34.5	——

			KLM	
K	60°	27′	39.5″	17056.6
L	70	54	5.5	18525.8
M	48	38	15	——
			KMN	
K	19	43	53.5	30560.4
M	75	36	40	31555.7
N	34	39	26.5	——
			KLN	
K	130	11	33	42562.7
L	34	29	42.5	——
N	15	18	44.5	——
			ELN	
E	6	6	39.43 *	——
L	152	15	25.15	186119
N	21	37	55.42 *	——
			ENP	
E	25	33	55.02 *	116660
N	110	55	29.83 *	252505.6
P	43	30	35.15 *	——
			ENS	
E	43	19	58.52	168327
N	87	30	29.58	245786
S	49	9	31.9	——
			NPS	
N	23	25	0.25	77237.2
P	119	41	41.64	——
S	36	53	18.11	——

In this register, each angle in the successive triangles is, for the sake of conciseness, marked by the single letter affixed to it, and the computed length of its opposite side in feet ranges in the same line. The addition of an asterisk denotes that an angle was not actually observed, but only deduced from calculation. The oblique triangles ABC and ABE have their sides BC and BE derived from other larger triangles, which were nearly equiangular. The triangles ELN and ENP had their angles discovered from conjoined observations. In general the several angles, as affected by the spherical excess, were corrected for computation by a sort of tentative process. It results from a train of calculations, that Dover Castle lies south 67° 44′ 34″ east,

and at the distance of 328231 feet or 62.165 miles, from Greenwich Observatory. On their part, the French astronomers, under the direction of Cassini, carried forward the trigonometrical operations from Dunkirk to Paris; employing Borda's *repeating circle*, an instrument much smaller and less perfect than Ramsden's theodolite, but formed on a principle which always procures the observer a near compensation of errors. From a comparison of the whole, it follows that the meridian of the Observatory of Paris lies 2° 19′ 51″ east from that of Greenwich, differing only nine seconds in defect from what the late Dr Maskelyne had previously determined from combined astronomical observations.

The success with which that great survey was attended, gave occasion both in France and England to still more extensive projects. The National Assembly, amidst other essential improvements which it meditated, having resolved to adopt a general and consistent system of measures, the length of a degree of the meridian at the middle point between the pole and the equator was proposed as a permanent basis. But to secure greater accuracy in determining the standard, it had been decided to prolong the observations on both sides of the mean latitude, and trace a chain of triangles over the whole extent from Dunkirk to Barcelona. This bold plan was executed in the course of the years 1792, 1793, 1794 and 1795, with equal sagacity and resolution, by MM. Delambre and Mechain, who, during all the horrors of revolutionary commotion, yet pressed forward their operations in spite of obstacles and dangers of the most sickening kind. After the various triangles, amounting in total to 115, had been observed, they were connected, in the neighbourhood of Paris, with a base of more than seven miles in length, and measuring, at the temperature of 16¼° on the centigrade scale, or 61¼° by Fahrenheit, 6075.9 toises from Melun to Lieursaint. A base of verification was likewise traced near the southern extremity of the line of survey, extending 6006.25 toises along the road from Perpignan to Narbonne. This base appeared not to differ one foot from the calculation founded on the other, though separated by a distance of 400 miles,—a convincing proof of the accuracy with which the observations had been made. A specimen of the French triangulation is given in the figure below, where the vertical line represents the meridian of Dunkirk, with the distances expressed by intervals of 10,000 toises.



A St Martin du Tërtre.
B Dammartin.
C Pantheon *at Paris.*
D Belle Assise.
E Brie.
F Montlheri.
G Lieursaint.
H Melun.
I Malvoisine.
K Torfou.
L Forêt.
M Chapelle.
N Pithiviers.
O Bois Commun.
P Chatillon.
Q Château-neuf.
R Orleans.
GH *The primary base.*

Calculation of the sides of the Triangles.

			ABC					FIG	
A	76°	2'	30''.66	17310.3013	F	49°	34'	22''.32	8369.1673
B	57	20	17.82	15017.3211	I	76	47	42.98	10703.5616
C	46	37	11.52	——	G	53	37	54.70	——
			BCD					IGH	
B	59	52	2.20	15756.8013	I	40	36	56.68	6075 8993
C	48	17	34.50	13601.3539	G	75	39	29.67	9042.5510
D	71	50	23.30	——	H	63	43	33.65	——
			CDE					FIK	
C	37	1	40.59	9516.5896	F	55	10	1.03	7357.8627
D	57	21	1.87	13305.8528	I	43	52	3.25	6212.1595
E	85	37	17.54	——	K	80	57	55.72	——
			CEF					IKL	
C	61	13	47.94	13101.0845	I	53	22	24.93	8349.1059
E	55	51	48.75	12370.8194	K	81	36	49.90	10292.0814
F	62	54	23.31	——	L	45	0	45.17	——
			EFI					ILM	
E	40	32	37.60	8852.8293	I	70	51	37.77	13438.2345
F	45	18	40.41	12374.2130	L	62	47	29.54	12650.5655
I	74	8	41.99	——	M	46	20	52.69	——

			LMN					OPQ	
L	68°	35′	59″.16	14402.0625	O	62°	31′	30″.34	10446.5520
M	51	5	13.26	12036.0949	P	93	0	17.27	11758.3955
N	60	18	47.58	——	Q	24	28	12.39	——
			MNO					PQR	
M	31	58	52.87	9190.1355	P	50	28	6.42	12053.9075
N	91	55	5.70	17341.8323	Q	87	35	8.93	15614.7105
O	56	6	1.43	——	R	41	56	44.65	——
			NOP						
N	31	53	2.40	4877.2386					
O	52	33	5.48	7330.6166					
P	95	33	52.12	——					

Through the whole process of their survey, the French astronomers have certainly displayed superior science. In deducing the correct results, they seem to exhaust all the refinements of calculation. The angles measured by the repeating circle, it was necessary to reduce, not only to the horizontal plane, but generally besides to the centre of observation. This would have required much nice and tedious computation; the labour of achieving such reductions was however greatly simplified and abridged, by help of concise *formulæ*, and the application of auxiliary tables. There is even room to suspect that those ingenious philosophers have carried the fondness for numerical operations to an excess, and often pushed the decimal places to a much greater length in their estimates than the nature of the observations themselves could safely warrant.

In the spring of 1799, the registers of all these operations were referred to a commission, consisting of the ablest members of the Institute, and some other learned men deputed from the countries then at peace with France. The various calculations were carefully examined and repeated; and a comparison of the celestial arc with that which had been measured in Peru having given $\frac{1}{334}$ for the oblateness of the earth, the length of the quadrant of the meridian, or the distance of the pole from the equator, was finally determined at 5130740 toises, the ten millionth part of which, or the space of 443.295936 lines forms the *metre*. This standard was afterwards definitively decreed by the Legislative Body.

Mechain, however, still anxious to realize his early project of extending the meridian as far as the Balearic Isles, again repaired to Spain, and conducted with incredible exertions a chain of triangles over the savage heights from Barcelona to Tortosa, and was about to observe the altitude of the stars, and measure the base of Oropesa, when, worn out by continued fatigue, he caught an epidemic fever, which fatally closed his meritorious labours, at Castellon de la Plana, in the kingdom of Valentia, about the latter part of September 1805.—The prosecution of the plan was subsequently committed to Biot, who has brought it to a fortunate conclusion. This ardent philosopher, during his stay on the rocky island of Formentera, had likewise an opportunity of making observations and experiments interesting to physical science. In the winter of 1806 and the spring of 1807, he continued the series of triangles from Barcelona to the kingdom of Valentia, and joined that coast with the Balearic Isles, by an immense triangle, of which one of the sides exceeded an hundred miles in length. At such prodigious distances, the stations, however elevated, and notwithstanding the fineness of the climate, could not be seen during the day; but they were rendered visible at night, by combining Argand lamps with powerful reflectors. These observations give a result which agrees almost exactly with what had been already found by Delambre and Mechain. If the mean were adopted, it would yet scarcely affect the length of the *metre* by the diminution of a four millionth part. The meridional arc extending from Dunkirk to Formentera, measures 12° 22′ 13″.395; and from this ample basis, the circumference of the earth is computed to be 24855.42 English miles.

In England, the prosecution of the trigonometrical survey, without aiming at such splendid views, has, suitably to the genius of the people, been directed to objects of more domestic interest, and perhaps real utility and importance. The perplexing inaccuracy of our best maps and charts had long been the subject of most serious complaint. It was in consequence resolved to extend the series of connected triangles over the whole surface of the island. But the death of General Roy, happening so early as 1790, threatened to prove fatal to the completion of his favourite scheme, and for which the talents and experience he possessed had so eminently fitted him. After some interruption, however, an opportunity was embraced of resuming that

noble plan; and it was, under the direction of the Board of Ordnance, committed to the care of Colonel Mudge, who, with equal ability and undiminished ardour, has, during the space of now almost twenty years, been engaged in carrying on the most extensive and varied system of operations ever attempted, and in a style of execution which reflects on him the highest credit. In 1793 and 1794, the chain of primary triangles was continued from Shooter's Hill to Dunnose in the Isle of Wight, including a great part of Surrey, Sussex, Hants, Wiltshire and Dorsetshire, and connecting with a new base of verification measured on Salisbury Plain. This base had, after correction, a length of 36574.4 feet, or 6.92697 miles, having lost almost a whole foot in being reduced from an elevation of 588 feet to the level of the sea. It differed scarcely an inch from the computation founded on the base of Hounslow Heath. In 1795, the triangles were carried into Devonshire; and they were continued in 1796 through Cornwall to the Scilly Islands. The West of England became the scene of repeated operations. In 1798, a third base was measured on King's Sedgemoor near Somerton, and found, after various corrections, to be 27680 feet, or 5.242425 miles, differing only about a foot from the result of the calculation dependent on that of Salisbury Plain. The survey now advanced to the centre of England, and was extended in 1803 to Clifton in Yorkshire; another base of verification, 26342.7 feet in length, having been measured at Misterton Carr, on the north of Lincolnshire. The triangles were next carried towards Wales, and made to rest on a base of 24514.26 feet, stretching from the western borders of Flintshire to Llandulas in Denbighshire. From this last base, numerous triangles have been extended in different directions; one series bending, through Anglesea and by Cardigan Bay, to the Bristol Channel; another penetrating into the central parts of England; while a third series stretches northwards, through Lancashire, Cumberland and Westmoreland, into Scotland, and uniting with the collateral chain of Misterton Carr from Yorkshire and Northumberland, is prolonged to the heights immediately beyond the Firth of Forth. We look forward with anxiety to the conclusion of this arduous undertaking. The mountains and islands near the western coast of Scotland will furnish triangles of vast extent. Colonel Mudge will not omit, we are confident, the opportunities that such stations may afford to determine the quantity of horizontal refraction,

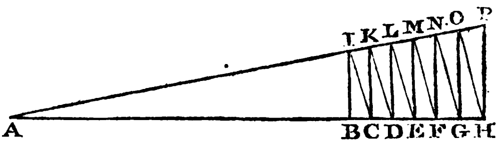
noting at the same time the variable state of the atmosphere. We have perfect reliance in the accuracy of his observations; yet it would be desirable in all cases, as in the French operations, that the third angle of each triangle were actually measured.

Besides the principal triangles thus determined, a multitude of subordinate ones were ascertained in the progress of the survey, and which serve to connect all the remarkable objects that occurred over the face of the country. The capital points were hence established for constructing the most accurate charts and provincial maps. A number of royal military surveyors, of approved skill, have since been constantly employed in filling up the secondary triangles, and embodying the skeleton plans. The various materials are *collected* at the drawing-room of the Tower, and there adjusted, reduced and combined. Under the same able direction, an extensive establishment has been formed in those spacious apartments, where a voluminous series of maps, and on the largest scale, are not only delineated but engraved. This truly national work advances with great activity, and has already proved highly advantageous to the public service. The Ordnance Maps, in elaborate accuracy, and even beauty of execution, surpass every thing hitherto designed.

To determine geometrically the altitude of a mountain requires, it hence appears, a nice operation performed with some large instrument. The barometrical mensuration of heights, is therefore, in most cases, preferred, as much easier and often more exact. This curious application was early suggested, by the objections themselves which ignorance opposed to Torricelli's immortal discovery of the weight of our atmosphere. But more than a century elapsed before the improvements in mechanics had completely adapted the machine to that purpose, and experiment combined with observation had ascertained the proper corrections. Barometers of various constructions are now made quite portable, and which indicate with the utmost precision the height of the mercurial column supported by the pressure of the atmosphere.

The air which invests our globe, being a fluid extremely compressible, must have its lower portions always rendered denser by the weight of the incumbent mass. To discover the law that connects

the densities with the heights in the atmosphere, it is only requisite, therefore, to apply the fact which experiment has established,—that the elasticity counterbalancing the pressure is exactly proportioned to the density. The elasticity of the air at any point of elevation, is hence measured by a column possessing the same uniform density, with a certain constant altitude. Let AB denote the height of this equiponderant column, and the perpendicular BI its density; and suppose the mass of air below to be distinguished into numerous *strata*, having each the same thickness BC. It is evident that the weight of the minute *stratum* at B will be expressed by BC; whence AB is to AC, or BI to CK, as the pressure at B to the augmented pressure at C, and therefore the density at C is denoted by CK. Again, having joined IC and drawn KD parallel, BI : CK : : BC : CD; and consequently CD will, on the same scale of density, express the weight of the stratum at C. Hence, AC is to CD, as CK to DL, or as the density at C is to that at D. It thus appears, that, repeating this process, the densities BI, CK, DL, &c. of the successive *strata* form a continued geometrical progression. But the same relation will evidently obtain at equal though sensible intervals. Thus, the density of the atmosphere is reduced nearly to one half, for every 3½ miles of perpendicular ascent. At 7 miles in height, the corresponding density is one-fourth; at 10½ miles, one-eighth; and at 14 miles, one-sixteenth.



The difference of altitude between two points in the atmosphere, is hence proportional to the difference of the logarithms of the corresponding densities or vertical pressures. But the heights of mountains may be computed from barometrical measurement to any degree of exactness, by a simple numerical approximation. Since AB, AC, AD, &c. are continued proportionals, it follows that AB : BC : : AB + AC + AD, &c. ; BC + CD + DE, &c. or BH. Let n denote the number of sections or *strata* contained in the mass of air, and $\frac{n}{2}$ (AB + AH) will nearly express the sum of the progression AB, AC, AD, &c.; wherefore, AB + AH :

BH : : 2AB : *n*BC, or the absolute difference of altitude. The height AB of the equiponderant column, reduced to the temperature of freezing water, is nearly 26,000 feet; and hence this general rule,— *As the sum of the mercurial columns is to their difference, so is the constant number* 52,000 *to the approximate height*. This number is the more easily remembered, from the division of the year into weeks.

Two corrections depending on the variation of temperature are besides required. Mercury expands about the 5,000th part of its bulk, for each degree of the centigrade scale; and hence the small addition to the upper column will be found, by removing the decimal point four places to the left, and multiplying by twice the *difference* of the attached thermometers. But the correction *afterwards* applied to the principal computation is of more consequence. Air has its volume increased by one 250th part, for each degree of heat on the same scale. If therefore the approximate height, having its decimal point shifted back three places, be multiplied by twice the sum of the degrees on the detached thermometers, the product will give the addition to be made.

An example will elucidate the whole process. In August 1775, General Roy observed the barometer on Caernarvon Quay at 30.091 inches, the attached thermometer being 15°.7, and the detached 15°.6 centigrade, while on the Peak of Snowdon the barometer stood at 26.409, the attached thermometer marking 10°.0, and the detached 8°.8. Here, twice the difference of the attached thermometers is 11°.4, which multiplied into .00264 gives .030, for the correction of the upper barometer. Next, 30.091 + 26.439 : 30.091 — 26.439, or 56.530 : 3.652 : : 52000 : 3359. Again, twice the sum of the degrees marked on the detached thermometers is 48.8, which multiplied into 3.359 gives 164; wherefore, the true height of Snowdon above the Quay of Caernarvon is 3359 + 164, or 3533 feet.

This mode of approximation may be deemed sufficiently near, for any heights which occur in this island; but greater accuracy is attained by assuming intermediate measures. To illustrate this, I shall select another example. At the very period when General Roy was making his barometrical observations at home, Sir George Shuckburgh Evelyn found the barometer to stand at 24.167 on the summit of the Mole, an insulated mountain near Geneva, the attach-

ed and detached thermometers indicating 14°.4 and 13°.4, while they marked 16°.3 and 17°.4 at a cabin below and only 672 feet above the lake, the altitude of the barometer at this station being 28.132. Now, $3.8 \times .0024 = .009$, and $24.167 + 009 = 24.176$; the arithmetical mean between which and 28.132 is 26.154; and hence, separately, 50.330 : 1.978 : : 52000 : 2044, and 54.286 : 1.978 : : 52000 : 1895. Wherefore, joining these two parts, 2044 + 1895, or 3939 expresses the approximate height. The final correction is $61.6 \times 3.939 = 243$, and consequently the Mole has its summit elevated 4854 feet above the lake of Geneva, and 6082 above the level of the sea.

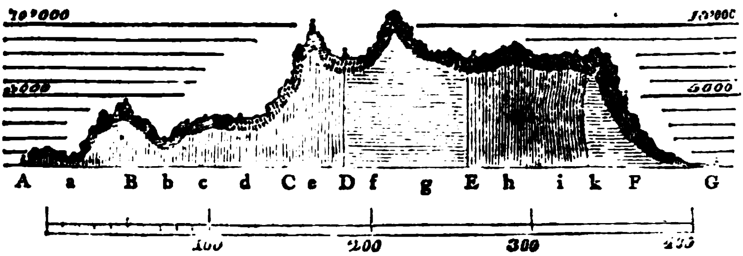
In general, let A and $A + nb$ denote the correct lengths of the columns of mercury at the upper and the lower stations; the approximate height of the mountain will be expressed by

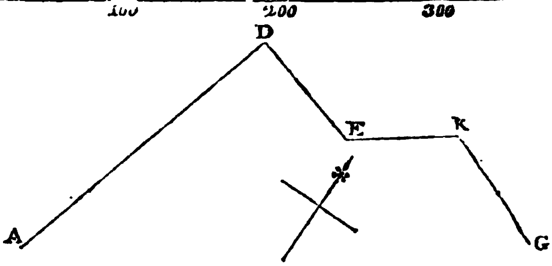
$$\left(\frac{b}{2A+b} + \frac{b}{2A+3b} + \frac{b}{2A+5b} \cdots + \frac{b}{2A+\overline{2n-1}.b}\right) 52000.$$

If n were assumed a large number, the result would approach to the accuracy of a logarithmic computation, though such an extreme degree of precision will be scarcely ever wanted.

To expedite the calculation of heights from barometrical observations, I have now caused Mr Cary, optician in London, to make for sale a sliding-rule, of an easy and commodious construction. That small instrument, which should be accompanied with a barometer of the lightest and most portable kind, will be found very useful to mineralogical travellers who have occasion to explore mountainous tracts. Nothing could tend more to correct our ideas of physical geography, than to have the principal heights in all countries measured, at least with some tolerable degree of precision. But the elevation of any place above the sea may be ascertained very nearly, from the comparison of even very distant barometrical observations, especially during the steadiness of the fine season in the happier climates. In this way, is traced a profile or vertical section, which exhibits at one glance the great features of a country. As a specimen, I have combined and reduced the sections which the celebrated philosophic traveller Humboldt has given of the continent of America, running in a twisted direction from Acapulco to Vera Cruz, and connecting the Pacific with the Atlantic Ocean.

A ACAPULCO.
a Peregrino.
B CHILPANSINGO.
b Mescala.
c Tepecuacuilco.
d Puente de Istla.
C CUERNAVACA.
e La cruz del Marques.
D MEXICO.
f Venta de Chalco.
g St Martin.
E LA PUEBLA DE LOS ANGELES.
h El Pinal.
i Perote.
k Cruz blanca.
F XALAPA.
G VERA CRUZ.





The divided scale expresses the horizontal distance in miles, while the parallels, on a much larger scale, mark the elevation in feet. This profile is really composed of four successive sections, which are distinguished by opposite shadings. The survey proceeded first along the road from Acapulco to Mexico, thence to Puebla de los Angeles, next to Cruz Blanca, and finally to Vera Cruz. These several directions and distances are expressed in the ground plan.

An attempt is likewise made in this profile, to convey some idea of the geological structure of the external crust:

Limestone, is represented by straight lines slightly inclined from the horizontal position.

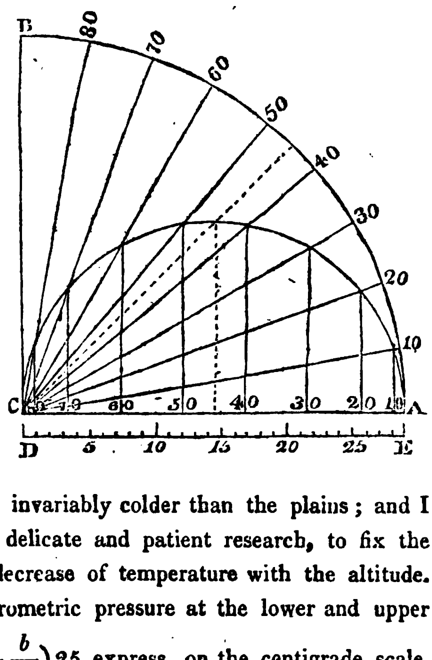
Basalt, by straight lines slightly reclined from the perpendicular.

Porphyry, by waved lines somewhat reclined.

Granite, by confused hatches.

Amygdaloid, by confused points.

But the easiest way of estimating within moderate limits the elevation of a country, is founded on the difference between the standard and the actual mean temperature as indicated by deep wells or copious and shaded springs. Professor Mayer of Göttingen, from a comparison of distant observations on the surface of the globe, proposed a *formula*, which, with a slight modification, appears to exhibit correctly the temperature of any place at the level of the sea. Let φ denote the latitude; and $29\ cos\varphi^2$, or $14\frac{1}{2}\ suvers\ 2\varphi$, will express, in degrees of the centigrade scale, the medium heat on the coast. But the gradations of climate are more easily conceived by help of a geometrical diagram. From the centre C, draw straight lines to the several degrees of the quadrant, and cutting the interior semicircle; then, the radius CA denoting 29°, the perpendiculars from the points of section will intercept segments proportional to the mean temperature expressed on DE.



The higher regions are invariably colder than the plains; and I have been able, after a delicate and patient research, to fix the law which connects the decrease of temperature with the altitude. If B and b denote the barometric pressure at the lower and upper stations; then will $\left(\frac{B}{b}-\frac{b}{B}\right)25$ express, on the centigrade scale, the diminution of heat in ascent. Hence, for any given latitude, that precise point of elevation may be found, at which eternal frost prevails. Put $x=\frac{b}{B}$, and $t=$ the standard temperature; then $\left(\frac{1}{x}-x\right)25=t$, or $x^2+.04tx=1$, which quadratic equation being resolved, gives the relative elasticity of the air at the limit of congelation, whence the corresponding height is determined. From these data the following table has been calculated.

Latitude.	Mean temperature at the level of the Sea. Centigrade.	Fahrenheit.	Height of Curve of Congelation Feet	Latitude.	Mean temperature at the level of the Sea. Centigrade.	Fahrenheit.	Height of Curve of Congelation. Feet.
0°	29°.00	84°.2	15207	46°	13°.99	57°.2	7402
1	28.99	84.2	15203	47	13.49	56.3	7133
2	28.96	84.1	15189	48	12.98	55.4	6865
3	28.92	84.0	15167	49	12.48	54.5	6599
4	28.86	83.9	15135	50	11.98	53.6	6334
5	28.78	83.8	15095	51	11.49	52.7	6070
6	28.68	83.6	15047	52	10.99	51.8	5808
7	28.57	83.4	14989	53	10.50	50.9	5548
8	28.44	83.2	14923	54	10.02	50.0	5290
9	28.29	82.9	14848	55	9.54	49.2	5034
10	28.13	82.6	14764	56	9.07	48.3	4782
11	27.94	82.3	14672	57	8.60	47.5	4534
12	27.75	82.0	14571	58	8.14	46.6	4291
13	27.53	81.6	14463	59	7.69	45.8	4052
14	27.30	81.1	14345	60	7.25	45.0	3818
15	27.06	80.7	14220	61	6.82	44.3	3589
16	26.80	80.2	14087	62	6.39	43.5	3365
17	26.52	79.7	13947	63	5.98	42.8	3145
18	26.23	79.2	13798	64	5.57	42.0	2930
19	25.93	78.7	13642	65	5.18	41.3	2722
20	25.61	78.1	13478	66	4.80	40.6	2520
21	25.28	77.5	13308	67	4.43	40.0	2325
22	24.93	76.9	13131	68	4.07	39.3	2136
23	24.57	76.2	12946	69	3.72	38.7	1953
24	24.20	75.6	12755	70	3.39	38.1	1778
25	23.82	74.9	12557	71	3.07	37.5	1611
26	23.43	74.2	12354	72	2.77	37.0	1451
27	23.02	73.6	12145	73	2.48	36.5	1298
28	22.61	72.7	11930	74	2.20	36.0	1153
29	22.18	71.9	11710	75	1.94	35.5	1016
30	21.75	71.1	11484	76	1.70	35.1	887
31	21.31	70.3	11253	77	1.47	34.6	767
32	20.86	69.5	11018	78	1.25	34.2	656
33	20.40	68.7	10778	79	1.06	33.9	552
34	19.93	67.9	10534	80	.87	33.6	457
35	19.46	67.0	10287	81	.71	33.3	371
36	18.98	66.2	10036	82	.56	33.1	294
37	18.50	65.3	9781	83	.43	32.8	226
38	18.01	64.4	9523	84	.32	32.6	167
39	17.51	63.5	9263	85	.22	32.4	117
40	17.02	62.6	9001	86	.14	32.3	76
41	16.52	61.7	8738	87	.08	32.2	44
42	16.02	60.8	8473	88	.04	32.1	20
43	15.51	59.9	8206	89	.01	32.0	5
44	15.01	59.0	7939	90	.00	32.0	0
45	14.50	58.1	7671				

This table will facilitate the approximation to the altitude of any place, which is inferred either from its mean temperature or its depth below the boundary of perpetual congelation. The decrements of heat at equal ascents are not altogether uniform, but advance more rapidly in the higher regions of the atmosphere. At moderate elevations, however, it will be sufficiently near the truth, to assume the law of equable progression, allowing in this climate one degree of cold by Fahrenheit's scale for every ninety yards of ascent. Thus, the temperatures of the Crawley and Black springs on the ridge of the Pentland hills, were observed by Mr Jardine, where they first issue from the ground, to be 46°.2 and 45°; which, compared with the standard temperature at the same parallel of latitude, would give 567 and 891 feet of elevation above the sea. The real heights found by levelling were respectively 564 and 882; a coincidence most surprising and satisfactory.—This ready mode of estimation claims especially the attention of agriculturists.

The rule stated above for computing the measurements by the barometer, seems to give results somewhat less, on the whole, than those which are obtained from geometrical observations. It would ensure greater accuracy, perhaps, to view the approximate height as answering to a temperature one degree under the point of congelation; and consequently, in applying the last correction, to add unit to the indications of the detached thermometers. But the whole subject demands a more thorough investigation. The elasticity of air is affected by moisture as well as heat, although the want of an exact instrument for measuring the former has hitherto prevented its influence from being distinctly noticed.

When the hygrometer which I have invented shall become better known to the public, it may not seem presumptuous to expect, in due time, more correct *data* concerning the modifications of the atmosphere. Yet, after all, in ascertaining the volume of a fluid subject to incessant fluctuation, it would be preposterous to look for that consummate harmony which belongs exclusively to astronomical science; nor can I help regarding the introduction of some late refinements into the *formulæ* for measuring heights by the barometer, and which would embrace the minutest anomalies of atmospheric pressure,—arising from the influence of centrifugal force, combined with the diminution of gravity in receding from the earth's centre,—as an utter waste of the powers of calculation.

I shall now subjoin a concise table of the most remarkable heights in different parts of the world, expressed in English feet. The altitudes measured by the barometer are marked B, while those derived from geometrical operations, and taken chiefly from the last work of Colonel Mudge, are distinguished by the letter G.

Snæ Fiall Jokul, *on the north-west point of Iceland*, -	4558 G
Hekla, *volcanic mountain in Iceland*, - - -	3950 G
Pap of Caithness, - - - -	1929
Ben Nevis, *Inverness-shire*, - - -	4380 B
Cairngorm, *Inverness-shire*, - -	4080 B
Ben Lawers, *Perthshire*, - - -	4015 B
Ben More, *Perthshire*, - - -	3870 B
Schihallien, *Perthshire*, - - -	3281 G
Ben Ledi, *Perthshire*, - - -	3009 B
Ben Lomond, *Stirlingshire*, - -	3240 B
Lomond Hills, east and west, *Fifeshire*, 1466 and	1721 G
Soutra Hill, *on the ridge of Lammer muir*, -	1716 G
Carnethy, *highest point of the Pentland ridge*, -	1700
Tintoc, *Lanarkshire*, - - - -	1720 B
Leadhills, *the house of the Director of the mines*, -	1564
Queensbery Hill, *Dumfries-shire*, - -	2259 G
Dunrigs, *Roxburghshire*, - - -	2408 G
Elden Hills, *near Melrose, Roxburghshire*, -	1364 G
Crif Fell, *near New Abbey in the Stewartry of Kirkcudbright*,	1831 G
Goat Fell, *in the Isle of Arran*, - -	2950 B
Paps of Jura, south and north, *in Argyllshire*, 2359 and	2470
Snea Fell, *in the Isle of Man*, - -	2004 G
Macgillicuddy's Reeks, *county of Kerry*, -	3404
Mourne Mountains, *county of Down*, - -	2500
Helvellyn, *Cumberland*, - - -	3055 G
Skiddaw, *Cumberland*, - - -	3022 G
Saddleback, *Cumberland*, - - -	2787 G
Whernside, *Yorkshire*, - - -	2384 G
Ingleborough, *Yorkshire*, - -	2361 G
Shunnor Fell, *Yorkshire*, - - -	2329 G
Snowdon, *Caernarvonshire*, - -	3571 G
Cader Idris, *Caernarvonshire*, - -	2914 G

Beacons of Brecknock,	- - -	2862 G
Plynlimmon, *Cardiganshire,*	- -	2463 G
Penmaen Mawr, *Caernarvonshire,*	- -	1540 G
Malvern Hills, *Worcestershire,*	- -	1444 G
Cawsand Beacon, *Devonshire,*	- -	1792 G
Rippin Tor, *Devonshire,*	- - -	1549 G
Brocken, *in the Hartz-forest, Hanover,*	-	3690
Schneekopf, *in Silesia,*	- - -	4950
Priel, *in Austria,*	- - -	6565
Peak of Lomnitz, *in the Carpathian ridge,*	-	8640
Mont Blanc, *Switzerland,*	- -	15646 G
Village of Chamouni, *below Mont Blanc,*	-	3367 G
Jungfrauhorn, *Switzerland,*	- -	13730
St Gothard, *Switzerland,*	- -	9075
Hospice of the Great St Bernard, *on the passage to Italy,*		8040 B
Village of St Pierre, *on the road to Great St Bernard,*		5338 B
Passage of Mont Cenis,	- - -	6778 B
Ortler Spitze, *in the Tyrol,*	- - -	15430
Rigiberg, *above the lake of Lucerne,*	- -	5408
Dole, *the highest point of the chain of Jura,*	-	5412 B
Mont Perdu, *in the Pyrenées,*	- - -	11283
Loneira, *in the department of the high Alps,*	-	14451
Peak of Arbizon, *in the department of the high Pyrenées,*		8344
Puy de Dome, *in Auvergne,*	- -	5197
Summit of Vaucluse, *near Avignon,*	- -	2150
Soracte, *near Rome,*	- - - -	2271 G
Monte Velino, *in the kingdom of Naples,*	-	8397 G
Mount Vesuvius, *volcanic mountain beside Naples,*	-	3978
Ætna, *volcanic mountain in Sicily,*	- -	10963 B
St Angelo, *in the Lipari Islands,*	- -	5260
Top of the Rock of Gibraltar,	- -	1439 B
Mount Athos, *in Rumelia,*	- -	3353
Diana's Peak, *in the Island of St Helena,*	-	2692
Peak of Teneriffe, *one of the Canary Islands,*	-	12358 B
Ruivo Peak, *the highest point in the Island of Madeira,*		5162
Table Mountain, *near the Cape of Good Hope,*	-	3520
Chain of Mount Ida, *beyond the plain of Troy,*	-	4960
Chain of Mount Olympus, *in Anatolia,*	-	6500

Italitzkoi, *in the Altaic chain,*	- -	10735
Awätsha, *volcanic mountain in Kamtchatka,*	-	9600
Taganai, *in the Uralian chain,*	- -	4912
The Volcano, *in the Isle of Bourbon,*	- -	7680
Ophir, *in the centre of the Island of Sumatra,*	-	13842
St Elias, *on the western coast of North America,*	-	12672
Chimborazo, *highest summit of the Andes,*	-	21440 B
Antisana, *volcanic mountain in the kingdom of Quito,*		19150 B
Cotopaxi, *volcanic mountain in the kingdom of Quito,*	-	18890 B
Tonguragua, *volcanic mountain, near Riobomba, in Quito,*		16579 B
Rucu de Pichincha, *in the kingdom of Quito,*	- -	15940 B
Heights of Assuay, *the ancient Peruvian road,*	-	15540 B
Peak of Orizaba, *volcanic mountain east from Mexico,*		17390 G
Lake of Toluca, *in the kingdom of Mexico,*	-	12195 B
City of Quito,	- - -	9560 B
City of Mexico,	- - - -	7476 B
Silla de Caraccas, *part of the chain of Venezuela,*		8640 B
Blue Mountains, *in the Island of Jamaica,*	-	7431
Pelée, *in the Island of Martinique,*	-	5100
Morne Garou, *in the Island of St Vincent's,*	-	5050
Mount Misery, *in the Island of St Christopher's,*	-	3711

I shall conclude with briefly stating the French measures. The Parisian foot was to the English, or the *toise* to the fathom, as 1.065777 to 1, or nearly as 16 to 15. The *metre*, or base of the new system, and equal to 39.371 English inches, ascends decimally, forming the *decametre* or *perch*, the *hectometre*, the *kilometre* or *mile*, and the *myriametre* or *league*, equivalent to 6.213856 of our miles; and descending by the same scale, it forms successively the *decimetre* or *palm*, the *centimetre* or *digit*, and the *millimetre* or *stroke*. The square of the *decametre* constitutes the *are*, and that of the *hectametre*, the *hectare* or *acre*, and equal to 2.47117 English acres. The cube of a *metre*, or 35.3171 feet, forms the unit of solid measure or the *stere*, that of a *decimetre*, or 61.028 inches forming the *litre* or *pint*; and the weight of this bulk of water at its greatest contraction makes the *kilogramme* or *pound*, equivalent to 2.1133 pounds Troy, the *gramme* answering to 15.444 grains.

FINIS.